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Contents

CHAPTER 5 Trigonometric Equations

CHAPTER 6 Oblique Triangles and Vectors

CHAPTER 7 Complex Numbers and Polar Coordinates

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Preface

Intent

This text is designed to serve as a one-semester introduction to trigonometry and its applications for college students.

Assumptions

It is assumed that students have basic skills in solving linear and quadratic equations, working with radicals, and simple graphing as well as some acquaintance with, and access to, a scientific calculator.

There is no separate chapter of review material. Most students react negatively to such a chapter, and many teachers get bogged down in unnecessary details in such a chapter. In this book, material is reviewed as it is encountered during the exposition of the new material. Focusing on old skills when they become necessary provides a more interesting sequence for students. The best motivation for reviewing something is the fact that it is needed to learn something new. Examples go out of their way to review the algebraic skills being applied, as they first arise.

Calculators

It is assumed that students have constant access to a modern scientific/engineering calculator. In terms of approximate calculations, the text is completely calculator oriented. However, the usual exact radian/degree values, for $\frac{\pi}{6}$(30°), $\frac{\pi}{4}$(45°), and their multiples, are explicitly used as well.

Many students today have access to graphing calculators. The book is designed to be used *without* these calculators, but their use is encouraged. The text shows how to use these calculators for the many graphs that occur in trigonometry. The TEXAS INSTRUMENTS TI-81 is used to illustrate within the text. Appendix B, *Further uses for graphing calculators and computers,* shows further detail on using the TI-81 and also presents material on using the CASIO *fx*-7000 graphing calculator.

Content

Chapter 1 reviews the basic geometric foundations of right triangle trigonometry, followed by the trigonometry of the right triangle. The term trigonometric *ratio* is used to distinguish the definitions made in terms of the right triangle from those made in chapter 2 in terms of points in the plane. We begin with right triangle trigonometry for at least the following four reasons. First, a survey of mathematics teachers indicated a clear preference for this approach. Second, this allows students to start the course in territory that is probably familiar, degree measure and right triangles. Third, applications in this area can be drawn from everyday experience. Fourth, this approach is most helpful to those students who may be starting a physics or other technical course at the same time.

The secant, cosecant, and cotangent functions are defined as reciprocals of the cosine, sine and tangent functions. In this way students are exposed to identities immediately, and, after all, this is the way we view these functions in advanced applications.

The use of the calculator, including a basic use of the inverse trigonometric functions, is covered in this chapter.

The chapter ends with a section that introduces trigonometric equations, both conditional and identities. It is important for students to see equations involving trigonometric functions early and throughout the course. The topic of conditional equations is revisited in the context of radian measure in chapter 2. In this way, the inevitable chapter on trigonometric equations (chapter 5) becomes simply an expansion of previous material, instead of the cold shock it often is for students.

Chapter 2 begins with a review of *functions,* which are introduced in terms of sets of ordered pairs. This approach combines mathematical precision with pedagogical simplicity (as opposed to a long statement about rules associating something with something else). It also permits a much simpler, more natural development of the concepts of one-to-one and inverse functions and has no negative impact on acquiring

ability with $f(x)$ terminology. In addition, this approach helps avoid the documented problem that most students cannot separate the concept of a function from a defining expression (i.e., an equation).

This chapter then covers the trigonometry of angles in standard position. In this exposition of analytic trigonometry, we refer to trigonometric *functions* not ratios. The topic is first completely covered with familiar angle measure. Radians are then introduced, and the topic is revisited in these terms, including the solution of simple conditional equations in terms of radians.

Chapter 3 gives a more in-depth treatment of the properties of the trigonometric functions as functions from the real numbers to the real numbers. Domains, ranges, and graphs are stressed. An uncommon method of graphing periodic functions is introduced here. Period and phase shift are not treated separately but are integrated into one step. It is our experience that this independently discovered the method of graphing quite complicated functions is easily and quickly taught and learned. It also provides a basis for understanding what makes periodic functions intrinsically different from nonperiodic functions.

This chapter also supports the use of graphing calculators to obtain graphs, for those who have access to these calculators.

Chapter 4 discusses the inverse of any function, then fully treats the inverse trigonometric functions. Students will have some feeling for these functions from earlier chapters, where they were explicitly mentioned in the context of finding unknown angles and solving conditional equations. The simplification of expressions involving these functions, using reference triangles, is stressed. This skill is very important in a calculus course.

The definitions of the inverse cosecant and secant functions are made in terms of the inverse sine and cosine functions. Although less common than other possible definitions, this is in keeping with other sources and provides a great simplicity in learning and using these functions. Some calculus texts define the ranges of these functions differently than as presented here (for good reason, in that context), but the additional complexity is not pedagogically warranted for an introduction. The definitions here, by the way, agree with those used in at least one popular symbolic algebra computer package.

Chapter 5 is a full treatment of trigonometric identities and conditional equations. The modern depth of coverage and the introduction in earlier chapters of most of the concepts should provide the basis for success for both teachers and students. Section 5–0 is explicitly designed to widen the scope of how students understand an expression to include trigonometric functions.

Chapter 6 presents topics that are applications of the previous material. The laws of sines and cosines are presented, and analytic vectors are introduced. The way in which the law of sines is presented should eliminate confusion for most students. The important fact that no triangle has more than one obtuse angle is stressed and used to solve triangles in a directed way.

Chapter 7 covers complex numbers, including De Moivre's theorem, and polar coordinates. The graphing of polar coordinates includes an independently discovered approach to the graphing of certain polar equations using the rectangular coordinate graph as a guide. This is a nontraditional approach, which the authors have used with success. It is easy to use, promotes a "feeling" for polar graphs, and reviews the graphing of trigonometric equations in rectangular coordinates.

Appendixes

Appendix A discusses using addition of ordinates for graphing. This topic is of much less practical value in the age of the electronic graphing device, but it can be a valuable pedagogical tool to promote the understanding of functions.

Appendix B presents more material on using graphing calculators. It can be used to enhance a course where every student has access to these wonderful devices.

Appendix C is the lengthy algebraic development of an identity, which may or may not be used in the course. This development has been set aside in an appendix because its inclusion within the text does not promote students' reading of the text or even a better understanding of the material.

Appendix D provides graphical material which the student may want to reproduce and have handy as an aid in studying.

Appendix E provides the answers to odd-numbered exercise problems, all chapter review problems, and all chapter test problems. Solutions are also provided for those trial exercise problems whose number is boxed in the exercise sets.

Exposition of the Material

We have attempted to write the exposition of the material in clear, understandable prose, in a logical order of development. Each section of a chapter is designed to provide accessible reading for students and clear examples of the skills that students are expected to master in that section. We have also tried to provide a cross section of *applications,* mainly in the exercises. These are designed to put the subject in a wider context of knowledge and therefore to pique students' interest. In particular, we have tried to show that trigonometry has become more important than ever in the age of the digital computer.

Each section of every chapter ends with a list of *mastery points,* which clearly states what students should know from that section. The exercise sets are designed to enable students to apply the material learned in the examples within the text, which attempt to provide a clear outline of the skills that the student must master. The exercise sets reinforce these skills explicitly, with many problems similar to the examples. Some problems require that students synthesize what has been learned, and a few require above-average efforts to solve. These are marked with the symbol . The complete solutions to those problems whose numbers are contained in boxes, called *trial problems,* are given in Appendix E.

A set of core problems are indicated in each exercise set by having their numbers in color. These problems exemplify each of the mastery points. Thus, if students can do these problems without error or difficulty then they have mastered the skills presented in that section. This is provided for those students who do not have the time to work a larger subset of the exercises. All students are still well advised to do the more difficult less skill oriented problems at the end of the exercise set when assigned.

Each chapter ends with a *chapter summary, chapter review,* and *chapter test.* The summary serves as a memory jogger, to be occasionally reviewed after the chapter is completed. The review consists of problems that enable students to practice the skills acquired throughout the chapter; these problems are keyed by section. The test is designed to allow students to practice a chapter test before seeing one in class. The material in the chapter test is removed from its exposition—this is the opportunity for students to see the material out of context, which is the situation when taking an in-class test.

Text Features

Some of the themes which we believe make this text special include the following.

- Algorithmic The text is explicit about procedures for accomplishing the skills required. These procedures are highlighted in the text. The examples explicitly follow these procedures.
- Detailed Skills and knowledge that are often assumed of students, but that in fact are not present, are explicitly covered. Rationalizing denominators is shown many times, as are many other algebraic manipulations, in the examples. Another example of this is the explicit coverage of the manipulation of the fractions involved in the radian measures, which are multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$, and achieving a feeling for the location on the unit circle of these measures. Still another is the algebraic manipulations often used in solving identities.
- Gradual skill building The level of algebraic competence required builds through the chapters. For example, solving equations in which a product equals zero is covered early, whereas factoring is postponed until the chapter on identities.

  Identities and conditional equations are difficult, so they are introduced early, and revisited lightly in the chapters preceding the chapter on trigonometric equations.

Solving right triangles is so basic to trigonometry at all levels of abstraction that it must be second nature. For this reason, right triangles are stressed not only in the applications and theoretical development but also as a means of finding the values of trigonometric functions when given the value of one of them. This is most important in a calculus course.

- Repetition of themes There are many themes that are revisited over and over in the chapters. Examples include solving conditional equations and manipulating identities, solving right triangles, and the idea of reference angles. In addition, a great deal of material is explicitly repeated when radian measure is covered in the last third of chapter 2.

  Vectors in two dimensions, complex numbers in polar form, and polar coordinates have a great deal of similarity in skills required, particularly in conversions between rectangular and polar forms. We have stressed these similarities.
- Balance of concrete versus abstract Most attention is paid to numeric manipulations, as in solving triangles and finding the values of trigonometric functions when given the value of one trigonometric function, but attention is paid to symbolic manipulation as well. Using these skills with symbolic manipulation is important in a calculus course, and they have not been short changed.

  Some of the exercises, always toward the end of each exercise set, also stress more abstract thinking, as well as the investigation of related ideas.

Changes from the Previous Edition

All references to tables of values have been deleted in this edition. This simplifies the material in the earlier chapters, allowing some material to migrate toward the front of the text. Functions are now explicitly reviewed earlier, at the beginning of chapter 2. This better provides the setting for the introduction of the trigonometric functions.

The topic of radian measure occurs earlier, in chapter 2 also. This provides the setting for reviewing the trigonometric functions in a new context.

A section on graphing by addition of ordinates is moved to the appendixes. Introductory material on the inverse trigonometric functions and trigonometric equations is more thoroughly integrated into the earlier chapters.

The chapter on identities is refashioned to include a more explicit review of equation solving and algebraic manipulation. Several sections were combined, which allows a more coherent presentation of the material.

In the previous edition, vectors and complex numbers each got two sections. The material on vectors was simplified by lessening the stress on geometric vectors and proceeding more quickly to analytic vectors. Thus, vectors are now treated in one section. It also seemed pedagogically appropriate and feasible to combine the two sections on complex numbers into one, with only minor adjustments in coverage, at no loss of depth. A section on graphing polar equations has been added.

The appendix on graphing calculators is new, as is the discussion of these calculators within the text. It is only a matter of time and economics before these devices are universally used in mathematics courses, and they can certainly be used with this text.

An appendix was added containing material students may want to reproduce xerographically. In particular, rectangular and polar coordinate templates are provided.

Supplements

For the instructor

The *Instructor's Manual* includes an introduction to the text, a guide to the supplements that accompany *Trigonometry,* and reproducible chapter tests. Also included are a complete listing of all mastery points and suggested course schedules based on the mastery points. The final section of the *Instructor's Manual* contains answers to the reproducible materials.

The *Instructor's Solutions Manual* contains completely worked-out solutions to all of the exercises in the textbook.

Selected *Overhead Transparencies* are available to enhance classroom presentations.

WCB Computerized Testing Program provides you with an easy-to-use computerized testing and grade management program. No programming experience is required to generate tests randomly, by objective, by section, or by selecting specific test items. In addition, test items can be edited and new test items can be added. Also included with the *WCB Computerized Testing Program* is an on-line testing option which allows students to take tests on the computer. Tests can then be graded and the scores forwarded to the grade-keeping portion of the program.

The *Test Item File* is a printed version of the computerized testing program that allows you to examine all of the prepared test items and choose test items based on chapter, section, or objective. The objectives are taken directly from *Trigonometry*.

For the student

The *Student's Solutions Manual* introduces the student to the textbook and includes solutions to every-other odd-numbered section exercise and odd-numbered end-of-chapter exercise problems. It is available for student purchase.

Videotapes covering the major topics in each chapter are available. Each concept is introduced with a real-world problem and is followed by careful explanation and worked-out examples using computer-generated graphics. These videos can be used in the math lab for remediation or even the classroom to motivate or enhance the lecture. The videotapes are available free to qualified adopters.

The concepts and skills developed in *Trigonometry* are reinforced through the interactive *Software*.

The Plotter is software for graphing and analyzing functions. This software simulates a graphing calculator on a PC. You may use it to do the technology exercises even if you don't have a graphics calculator. A manual is included that describes operations and includes student exercises. The software is menu driven and has an easy-to-use window-type interface. The high-quality graphics can also be used for classroom presentation and demonstration. Students who go on to calculus classes will want to keep the software for future use.

Acknowledgments

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Throughout the development, writing, and production of this text, two WCB employees have been of such great value they deserve special recognition: Theresa Grutz and Eugenia M. Collins.

CHAPTER 1

The Trigonometric Ratios

1–0 Introduction

Initially trigonometry was developed to express relationships between the sizes of the arcs in circles and the chords determining those arcs. (An arc is a portion of the circumference of a circle; a chord is a line segment going from one point on a circle to another.) These relationships were used in astronomy more than two thousand years ago to study what is called the celestial sphere. In fact, until the fifteenth century, trigonometry was mostly applied to spheres. This part of trigonometry is now called spherical trigonometry, and it is still used in navigation and astronomy.

After the fifteenth century, trigonometry was also used to relate the measure of angles in a triangle to the lengths of the sides of the triangle. The word ''trigonometry,'' which means ''triangle measurement,'' is credited to Bartholomaus Pitiscus (1561–1613). Besides being used in surveying, trigonometry became important for the physics being developed by Sir Isaac Newton and others. Practically all the ideas of trigonometry had been developed by the eighteenth century.

Over the next two hundred years, trigonometry became more and more important as it was used to describe many physical phenomena, such as electricity, magnetism, and sound.

Today, in the age of the computer, trigonometry is being used even more. The computer can control manufacturing machines to great precison, but only if trigonometry is used to describe where this precision is to be applied. This whole area of engineering, called numerical control, relies heavily on the ideas we will study in this book. Computer graphics use aspects of projective geometry, which again uses the concepts of trigonometry. Voice recognition by computers uses a concept in mathematics called Fourier transforms, which again is built on these same ideas.

From medicine to manufacturing, from solar design to computer art, the ideas we study here are found over and over again.

In this chapter we learn how trigonometry provides a method for telling us a great deal about a right triangle from limited information. These simple ideas, widely applied in science, engineering, and mathematics, lay the groundwork for some of the more advanced ideas we will eventually need.

Students are expected to have access to engineering or scientific calculators to facilitate many of the calculations throughout this text. Many calculations are followed by the notation $\boxed{\text{CS } n}$, where n is an integer. This indicates that the appropriate calculator steps are shown at the end of that section. Steps are shown for a generic algebraic calculator (one with an $\boxed{=}$ key, indicated by Ⓐ) and for a generic postfix notation calculator (indicated by Ⓟ). The latter calculator has no $\boxed{=}$ key but has an $\boxed{\text{ENTER}}$ or $\boxed{\uparrow}$ key instead. Steps are also shown for the Texas Instruments TI-81 calculator. These are indicated by $\boxed{\text{TI-81}}$.

Some preliminary words are in order about calculating with a calculator. It is difficult to give a general rule for the number of digits that should appear in the final result of a calculation, and a discussion of this is outside the scope and intent of this book. The number of digits to which an approximate number should be rounded is specified throughout the text. The number of digits is chosen to represent a reasonable, if sometimes arbitrary, value.

Also, we will apply the most straightforward rounding rule to values. That is, if the first discarded digit is 5 or above we round up, otherwise we do not. Thus, 2.349 rounds to 2.35 to two decimal places, and to 2.3 to one decimal place.

1–1 Angle measurement, the right triangle, and the Pythagorean theorem

Angle measurement

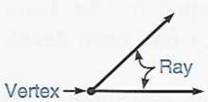


Figure 1–1

An **angle** is composed of two rays, both beginning at what is called the **vertex** of the angle. Figure 1–1 shows a representation of an angle.

In modern geometry a method of angle measurement is assumed. We will assume that angles can be measured in **degrees.** The notation for degrees is °. A common and useful interpretation of angle measure is "the amount of rotation" of one ray away from the other. In this context, 90° corresponds to a quarter-rotation, 180° to a half-rotation, 270° to a three-quarter rotation, and 360° to a full rotation. Naturally, the measurement of an angle does not necessarily imply any actual rotation. See figure 1–2.

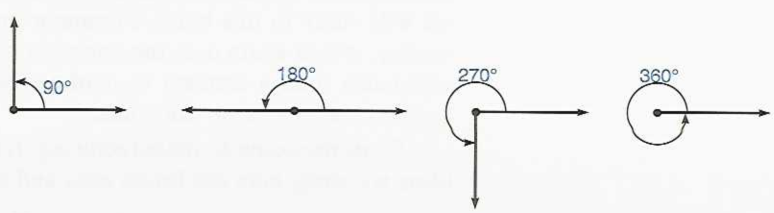


Figure 1–2

An angle with measure between 0° and 90° is said to be **acute;** an angle with measure 90° is **right;** an angle with measure between 90° and 180° is **obtuse;** and an angle with measure 180° is a **straight angle.**

One degree is divided into smaller units in two ways: using the degree, minute, second system and using the decimal degree system.

In the **degree, minute, second (DMS) system** a degree is divided into 60 equal parts called *minutes,* and each minute is divided into 60 equal parts called *seconds.* This is analogous to the way in which hours are divided into minutes and seconds on the clock. The notation for minutes is ′ and the notation for seconds is ″. For example, 51°18′22″ means 51 degrees, 18 minutes, and 22 seconds.

In the **decimal degree system** the degree measure is written in decimal notation. For example, 2.53° is in decimal degrees and means 2 and 53 hundredths of a degree.

Calculators will generally not perform operations on values in the DMS system, so we must be able to convert from this system to decimal form. Many calculators are programmed to do this. They typically use keys marked $\boxed{\circ\ ,\ ,,}$ or $\boxed{\to\text{H}}$. This is illustrated in example 1–1 A for two typical calculators. Also note that we use the fact that there are 60 minutes in one degree, and $60^2 = 3{,}600$ seconds in one degree.

■ ***Example 1–1 A***

Convert 46°42′27″ to decimal degrees to the nearest 0.001°.

Manually:

$$46° + \left(\frac{42}{60}\right)^{\circ} + \left(\frac{27}{3600}\right)^{\circ}$$ Rewrite minutes and seconds as fractional parts of a degree

46.7075° $\boxed{\text{CS 1}}$

46.708° Round to the nearest 0.001°

Calculator A:

46 $\boxed{\circ\ ,\ ,,}$ 42 $\boxed{\circ\ ,\ ,,}$ 27 $\boxed{\circ\ ,\ ,,}$

Calculator B:

46.4227 $\boxed{\to\text{H}}$ "H" stands for hours ■

The right triangle

A closed figure composed of three straight sides will always include three angles, and these figures are therefore called **triangles**[1] (*tri* is from the Latin for three). (Actually the angles are formed by extending the sides, which are line segments, to form rays.)

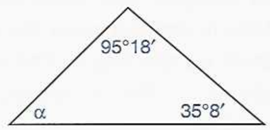
A useful property of triangles is that *the measures of the three angles of a triangle always add up to 180°*. This fact was known thousands of years ago and its first known demonstration appears in Euclid's *Elements,* an ancient Greek book on geometry. This allows us to find the measure of the third angle in a triangle if we know the measures of the other two: if we add the measures

[1]Note that, unless otherwise specified, all references to lines, triangles, etc., are to these concepts as they occur in plane, or Euclidean, geometry. Fortunately, this is the geometry with which the reader is most likely to be familiar, but it is worth mentioning that mathematics recognizes other ''types'' of geometry, called non-Euclidean geometries. In fact, several of these geometries find applications in modern physics.

of the two known angles and subtract this from the known total, 180°, the result must be the measure of the third angle.

Note We often denote angles using Greek letters. The letters most often used are α (alpha), β (beta), γ (gamma), and θ (theta).

■ ***Example 1–1 B*** Find the measure of angle α in the triangle in the figure.



$$\begin{aligned}\alpha &= 180° - (95°18' + 35°8') \\ &= 180° - 130°26' = 179°60' - 130°26' = 49°34'\end{aligned}$$ ■

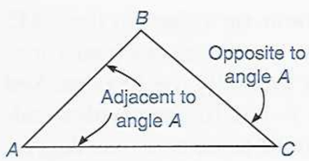


Figure 1–3

When we say that a side of a triangle is **opposite** to an angle, we are referring to the side that is not used to form the angle. When we say that a side of a triangle is **adjacent** to an angle, we mean that the side actually forms one side of the angle. For example, in triangle ABC in figure 1–3, we would say that side BC is opposite to angle A, while sides AB and AC are each adjacent to angle A.

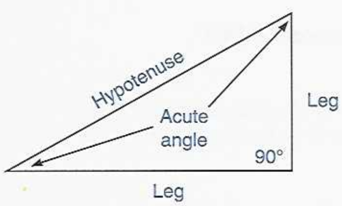


Figure 1–4

A **right triangle** is a triangle in which one of the angles is a right angle (90°). In such a triangle, two of the angles must be acute (less than 90°), since their measures must add up to 90°. The side of a right triangle that is opposite the right angle is called the **hypotenuse,** and the sides that are adjacent to the right angle are sometimes called **legs.** See figure 1–4.

One particular way to label right triangles is widely used. Unless we are told otherwise, in a right triangle we always label the right angle C and the two acute angles A and B. The lengths of the legs are always labeled a and b, with a opposite angle A and b opposite angle B. The hypotenuse is always labeled c. This is illustrated in figure 1–5.

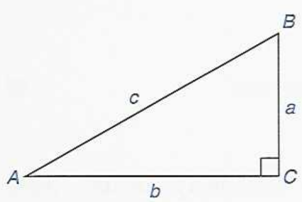


Figure 1–5

Note The symbol ∟ denotes a right angle.

The Pythagorean theorem

We often use the following theorem.[2] It is one of the most important facts in mathematics.

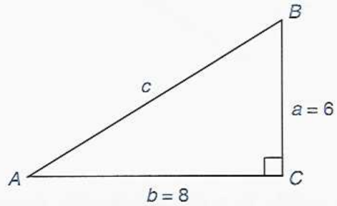
> **The Pythagorean theorem**
> In a right triangle with legs having lengths a and b and hypotenuse having length c,
> $$a^2 + b^2 = c^2$$

We use the Pythagorean theorem to find the length of the hypotenuse of a right triangle if we know the lengths of the two legs.

[2] The word *theorem* means a statement that has been proved to be true, and the proof of this theorem is credited to the Greek mathematician Pythagoras (sixth century B.C.), who is said to have sacrificed an ox as an offering of thanks. In the last two thousand years literally hundreds of proofs of this theorem have been given.

■ Example 1–1 C

If the two legs of a right triangle have lengths 6 and 8, what is the length of the hypotenuse?



To use the Pythagorean theorem, we label one leg a and the other b. See the figure. Let $a = 6$ and $b = 8$ and use the theorem as follows:

$$a^2 + b^2 = c^2 \text{ so } 6^2 + 8^2 = c^2$$
$$100 = c^2$$

To find c we take the principal square root of 100:

$$\sqrt{100} = c$$
$$10 = c$$

The Pythagorean theorem also provides a way to find the length of one leg of a right triangle if the hypotenuse and the other leg are known.

■ Example 1–1 D

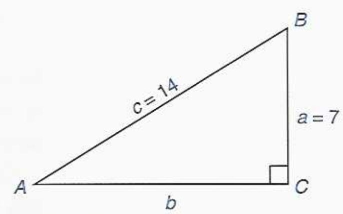
If one leg of a right triangle has length 7, and the hypotenuse has length 14, find the length of the other leg. Find the answer both exactly and to the nearest tenth.

Let $a = 7$ and b be the unknown side, as in the figure. Then,

$$a^2 + b^2 = c^2$$

so

$$7^2 + b^2 = 14^2$$
$$49 + b^2 = 196$$
$$b^2 = 147$$
$$b = \sqrt{147}$$
$$b = \sqrt{(49)(3)}$$
$$b = \sqrt{49}\sqrt{3}$$
$$b = 7\sqrt{3} \quad \text{Exact solution}$$
$$b \approx 12.1 \quad \text{Approximate solution}^3 \ \boxed{\text{CS 2}}$$

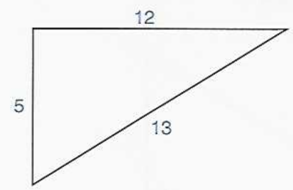


It can also be shown that *if* $a^2 + b^2 = c^2$, *then that triangle is a right triangle, and the right angle is opposite side c.*

[3]When we write something like $7\sqrt{3} \approx 12.1$, we mean that $7\sqrt{3}$ is *approximately* 12.1. This use of the symbol ≈ is adhered to in this text to signify approximate values.

■ **Example 1–1 E**

1. Is the triangle in the figure a right triangle?



If the triangle is a right triangle, then the hypotenuse would be the side having length 13 (*the hypotenuse of a right triangle is always the longest side*). Therefore the legs would have lengths 5 and 12. We now add the squares of the lengths of the legs and see if this sum equals the square of the hypotenuse, 13.

$$5^2 + 12^2 = 25 + 144 = 169, \text{ and}$$
$$13^2 = 169$$

Thus, $a^2 + b^2 = c^2$ and, therefore, the triangle is a right triangle.

Note In part 1 of example 1–1 E, we stated that the hypotenuse of a right triangle is always the longest side. This is because *in any triangle*, whether or not it is a right triangle, *the longest side is always opposite the largest angle.* Similarly, the shortest side is always opposite the smallest angle.

2. The sides of a triangle have lengths 32, 53, and 62. Is the triangle a right triangle?

Using the Pythagorean theorem,

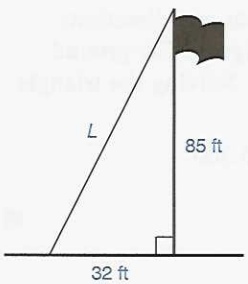
$$32^2 + 53^2 = 1{,}024 + 2{,}809 = 3{,}833 \text{ and } 62^2 = 3{,}844$$

Since $a^2 + b^2 \neq c^2$, we see that the triangle is not a right triangle. ■

As we said earlier, the Pythagorean theorem is one of the most important facts in mathematics because it has so many applications to practical situations and technical problems. Many situations in science and technology can be described, or modeled, using right triangles. If we know that a physical situation can be described in terms of right triangles, then all of the mathematics that applies to right triangles can be used to learn more about the situation or to solve a given problem.

In practice we must often make simplifying assumptions about the situation. For example, we may assume that a telephone pole makes a right angle with the ground, when in fact it is unlikely that the angle is exactly 90°, or we may assume that the earth is flat, that ''level'' roads are perfectly level, etc.

■ Example 1–1 F



1. A guy wire is to be attached to the top of a flagpole and anchored in the ground at a point 32 feet from the base of the flagpole. If the pole is 85 feet high, how long will the guy wire have to be, to the nearest foot?

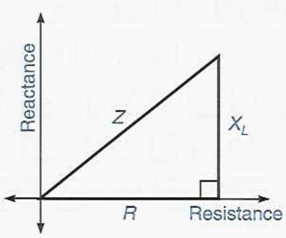
We first draw a figure to describe the problem, as shown. We assume that a flagpole is constructed to form as close to a 90° angle with the ground as possible. Also, we ignore the fact that the wire will actually sag somewhat and therefore is not a true straight line.

We can see from the figure that the unknown length of the guy wire L is the hypotenuse of a right triangle in which the legs have lengths 85 and 32. Thus, the Pythagorean theorem provides the answer.

$$85^2 + 32^2 = L^2$$
$$8{,}249 = L^2$$
$$\sqrt{8{,}249} = L$$
$$90.8 \approx L \qquad \boxed{\text{CS 3}}$$

To the nearest foot, the guy wire must be 91 feet long.

2. In the theory of alternating current[4] in electronics, the total **circuit impedance** Z in an inductive circuit can be found if we know the **inductive reactance** X_L and the **resistance** R by using the impedance diagram shown in the figure.

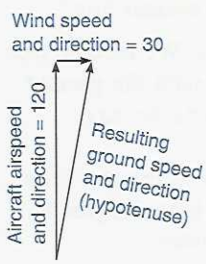


Suppose $R = 320$ ohms and $X_L = 160$ ohms. Find Z to the nearest ten ohms.

Using the Pythagorean theorem,

$$Z^2 = R^2 + X_L^2$$
$$Z^2 = 320^2 + 160^2$$
$$Z^2 = 128{,}000$$
$$Z = \sqrt{128{,}000}$$
$$Z \approx 357.8$$
$$Z \approx 360 \text{ ohms, to the nearest ten ohms}$$

[4]As in many other examples and problems in this text, the reader may not be familiar with the terminology and/or the facts involved in the application being illustrated. Please note, however, that where the necessary background is not provided, the problem is phrased so that it is clear to the reader which mathematical operation is required to achieve the desired result.



3. An aircraft is flying with an airspeed of 120 knots[5] and a heading of due north. A 30-knot wind is blowing from the west. What is the aircraft's ground speed, to the nearest knot?

It is a fact of physics that we can represent the speeds and directions given above in a right triangle as illustrated in the figure. The ground speed of the aircraft is the length of the hypotenuse. Solving the triangle shows that the ground speed is 124 knots:

$$\begin{aligned}(\text{ground speed})^2 &= 120^2 + 30^2 = 15{,}300\\ \text{ground speed} &= \sqrt{15{,}300}\\ &\approx 124\end{aligned}$$

Calculator steps

1. A 46 $\boxed{+}$ 42 $\boxed{\div}$ 60 $\boxed{+}$ 27 $\boxed{\div}$ 3600 $\boxed{=}$ Display $\boxed{46.7075}$

P 46 $\boxed{\text{ENTER}}$ 42 $\boxed{\text{ENTER}}$ 60 $\boxed{\div}$ $\boxed{+}$ 27 $\boxed{\text{ENTER}}$ 3600 $\boxed{\div}$ $\boxed{+}$

$\boxed{\text{TI-81}}$ 46 $\boxed{+}$ 42 $\boxed{\div}$ 60 $\boxed{+}$ 27 $\boxed{\div}$ 3600 $\boxed{\text{ENTER}}$

2. A 7 $\boxed{\times}$ 3 $\boxed{\sqrt{x}}$ $\boxed{=}$ Display $\boxed{12.12435566}$

P 7 $\boxed{\text{ENTER}}$ 3 $\boxed{\sqrt{x}}$ $\boxed{\times}$

$\boxed{\text{TI-81}}$ 7 $\boxed{\times}$ $\boxed{\sqrt{\ }}$ 3 $\boxed{\text{ENTER}}$

3. A 85 $\boxed{x^2}$ $\boxed{+}$ 32 $\boxed{x^2}$ $\boxed{=}$ $\boxed{\sqrt{x}}$ Display $\boxed{90.82400564}$

P 85 $\boxed{x^2}$ 32 $\boxed{x^2}$ $\boxed{+}$ $\boxed{\sqrt{x}}$

$\boxed{\text{TI-81}}$ $\boxed{\sqrt{\ }}$ $\boxed{(}$ 85 $\boxed{x^2}$ $\boxed{+}$ 32 $\boxed{x^2}$ $\boxed{)}$ $\boxed{\text{ENTER}}$

Mastery points

Can you

- State whether a given angle is acute, right, straight, or obtuse?
- Convert from the degree, minute, second system to the decimal degree system?
- Find the third angle of any triangle when you know the other two angles?
- Give the definition of right triangle and draw and label one using the conventional notation?
- State the Pythagorean theorem?
- Use the Pythagorean theorem to find one side of a right triangle when you know the other two sides?

[5] One knot means one nautical mile per hour. A nautical mile is $\frac{1}{60}$ of a degree on the earth's circumference. It is approximately 6,080.27 feet. Nautical miles are often used in the navigation of ships and aircraft.

Exercise 1–1

Convert each angle to its measure in decimal degrees. Round the answer to the nearest 0.001° where necessary. Also, state whether each angle is acute, obtuse, straight, or right.

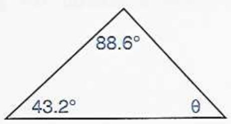
1. 13°25′	**2.** 111°56′	**3.** 0°12′	**4.** 42°37′
5. 25°33′19″	**6.** 87°2′13″	**7.** 165°47′	**8.** 19°15′
9. 33°5′55″	**10.** 0°19′12″	**11.** 159°59′	**12.** 20°1′

13. Draw a representation of a right triangle and label it using the conventional labeling (using *A*, *B*, *C*, *a*, *b*, *c*).

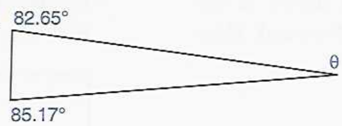
14. State the Pythagorean theorem from memory.

In the following problems find the measure of the angle θ.

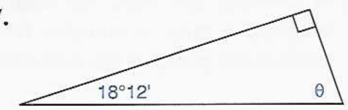
15.



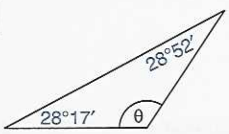
16.



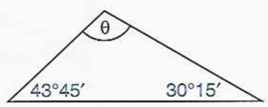
17.



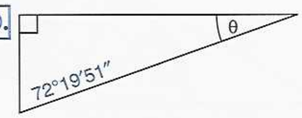
18.



19.

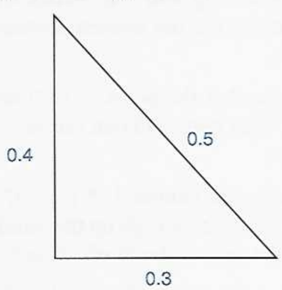


20.

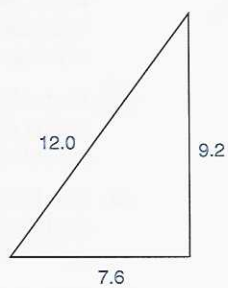


In the following problems state whether each triangle is a right triangle or not; if the triangle is a right triangle, state the length of the hypotenuse.

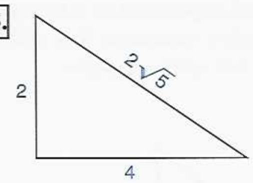
21.



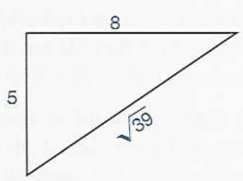
22.



23.

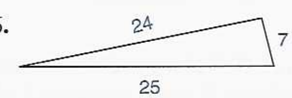


24.



25.

26.



In the following problems two of the three sides of a right triangle are given. Use the Pythagorean theorem to calculate the length of the missing side; leave your answer in exact form and also approximate your answer to the same number of decimal places as the data (if necessary).

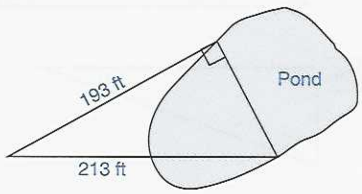
	a	*b*	*c*		*a*	*b*	*c*		*a*	*b*	*c*
27.	9	12	?	**28.**	10	?	26	**29.**	?	8	10
30.	5	10	?	**31.**	12	?	18	**32.**	?	6.8	9.2
33.	$\sqrt{5}$	3	?	**34.**	13.2	19.6	?	**35.**	100	150	?
36.	0.66	1.42	?	**37.**	$\sqrt{7}$	3	?	**38.**	4	?	$\sqrt{23}$
39.	6.3	?	15.0	**40.**	2	?	3	**41.**	$3\sqrt{2}$	$4\sqrt{5}$	?
42.	$3\sqrt{2}$	?	$4\sqrt{5}$	**43.**	?	19	28	**44.**	1,002	3,512	?
45.	1	1	?	**46.**	30	?	50				

Solve the following problems.

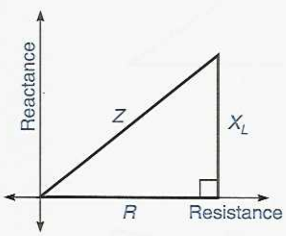
47. A flagpole is 55 feet tall and is supported by a wire attached at the top of the pole and to the ground 26 feet from the base of the pole. How long is the wire, to the nearest foot?

48. A flagpole is 93 feet tall and is supported by a guy wire that is 157 feet long, attached at the top of the pole and to the ground some distance from the base of the pole. Find the distance of the wire's ground attachment point from the base of the pole, to the nearest foot.

49. A surveyor has made the measurements shown in the diagram in order to compute the width of a pond. How wide is the pond, to the nearest 0.1 foot?



This diagram is called an impedance diagram. It is used to compute total impedance in a certain electronic circuit. X_L means inductive reactance, R is resistance, and Z is impedance. All units are ohms.



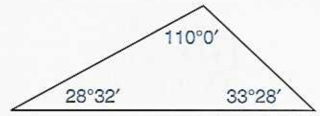
50. If X_L is 40.0 ohms and R is 56.6 ohms, calculate the total impedance Z to the nearest 0.1 ohm.

51. Suppose that X_L is 5.68 ohms and R is 19.25 ohms. Find Z to the nearest 0.01 ohm.

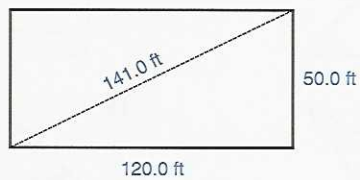
52. If $Z = 213$ ohms and $R = 183$ ohms, find X_L to the nearest ohm.

53. If $X_L = 2{,}150$ ohms and $Z = 4{,}340$ ohms, find R to the nearest 10 ohms.

54. A triangular piece of land has been surveyed and the results are shown in the diagram. What can you say about the accuracy of the survey? Give a reason for your answer.



55. A rectangular piece of land has been surveyed and the results are shown in the diagram. What can you say about the accuracy of the survey? Give a reason for your answer.



56. A machinist has to cut a rectangular piece of steel along its diagonal. The saw that will be used can cut this type and thickness of steel at the rate of 0.75 inch per minute. If the piece is 13.8 inches long and 9.6 inches wide, calculate how many minutes (to the nearest minute) it will take to cut the piece.

57. Do problem 56 but assume that the piece is 15.0 centimeters (cm) long and 10.5 cm wide and that the saw will cut at 0.8 cm per minute.

58. The ladder on a fire truck can extend 125 feet. If the truck is 25 feet from a building, how high up the building can the ladder reach, to the nearest tenth of a foot?

59. If the fire truck of problem 58 moves 5 more feet from the building (to 30 feet), does the height up the building that the ladder can reach decrease by 5 feet? If not, how much does it decrease, to the nearest tenth of a foot?

60. Find the ground speed, to the nearest knot, of an aircraft flying with a heading due east and an airspeed of 132 knots if there is a wind blowing from the north at 23 knots.

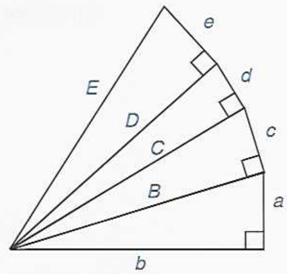
61. Find the ground speed, to the nearest knot, of an aircraft flying with a heading due west and an airspeed of 105 knots if there is a wind blowing from the north at 18 knots.

62. Find the ground speed, to the nearest knot, of an aircraft flying with a heading due south and an airspeed of 178 knots if there is a wind blowing from the east at 25 knots.

63. Find the speed relative to the ground, to the nearest knot, of a boat heading directly across a river at 16 knots if the current is moving at 4.3 knots.

64. To the nearest $\frac{1}{4}$ inch, find the length of the diagonal of an $8\frac{1}{2}$-inch by 11-inch piece of paper.

65. In the "Mathematics of Warfare" by F. W. Lanchester (from *The World of Mathematics* by James R. Newman), Mr. Lanchester presents the idea that, all other things being equal, the strengths of fighting forces add in a manner proportional to the squares of their numbers. Referring to the diagram, this means that a force of size *B* is equal to two forces of sizes *a* and *b;* that *C* is equal to the combined strengths of *a, b,* and *c;* etc. Assuming that forces *a, b, c, d,* and *e* are of size 20, 5, 12, 8, and 10, respectively, find the size of force *E* that is equivalent to the combined strengths of these forces. Find this force to the nearest unit.



1–2 *The trigonometric ratios*

Trigonometry has existed in one form or another for more than two thousand years. The creator of trigonometry is said to have been the Greek Hipparchus of the second century B.C. The Hindus and, primarily, the Arabs continued developing the subject. In the fifteenth and sixteenth centuries, the Germans developed trigonometry into the form presented here.

The primary trigonometric ratios

We first define the **sine, cosine,** and **tangent** ratios, abbreviated sin, cos, and tan, respectively. We refer to these as the primary trigonometric ratios.

Primary trigonometric ratios

If θ is either of the two acute angles in a right triangle (refer to figure 1–6), then:

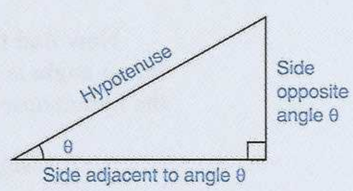


Figure 1–6

Ratio	**Definition**
$\sin\theta =$	$\dfrac{\text{length of side opposite } \theta}{\text{length of hypotenuse}}$
$\cos\theta =$	$\dfrac{\text{length of side adjacent to } \theta}{\text{length of hypotenuse}}$
$\tan\theta =$	$\dfrac{\text{length of side opposite } \theta}{\text{length of side adjacent to } \theta}$

Note "sin θ" means "the sine of angle θ" and is read "sine theta." "cos θ" means "the cosine of angle θ" and is read "cosine theta." "tan θ" means "the tangent of angle θ" and is read "tangent theta."

An abbreviated version of these definitions is

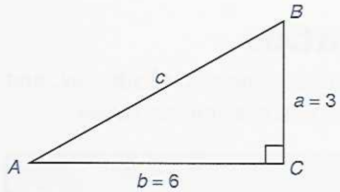
$$\sin\theta = \frac{\text{opp}}{\text{hyp}}$$
$$\cos\theta = \frac{\text{adj}}{\text{hyp}}$$
$$\tan\theta = \frac{\text{opp}}{\text{adj}}$$

These ratios are used in astronomy, surveying, engineering, science, and mathematics. In fact, there is virtually no area of science and technology that does not use them.

■ *Example 1–2 A*

1. Find the sine, cosine, and tangent ratios for angles A and B in right triangle ABC, where $a = 3$ and $b = 6$.

   Recall that the right angle is always labeled C, side a is opposite angle A, and side b is opposite angle B. We show this and the given data in the figure.



   First, we find c by the Pythagorean theorem.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 3^2 + 6^2 &= c^2 \\ 45 &= c^2 \\ \sqrt{45} &= c \\ \sqrt{(9)(5)} &= c \\ \sqrt{9}\sqrt{5} &= c \\ 3\sqrt{5} &= c \end{aligned}$$

   Now find the sine ratio for each angle A and B. Since the sine ratio for an angle is the length of the side opposite the angle over the length of the hypotenuse, we have

$$\sin A = \frac{\text{side opposite angle } A}{\text{hypotenuse}} = \frac{a}{c} = \frac{3}{3\sqrt{5}} = \frac{1}{\sqrt{5}}$$
$$= \frac{1}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$
$$\sin B = \frac{\text{side opposite angle } B}{\text{hypotenuse}} = \frac{b}{c} = \frac{6}{3\sqrt{5}} = \frac{2}{\sqrt{5}}$$
$$= \frac{2}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

To find the cosine ratio we form the ratios of the sides adjacent to each angle over the length of the hypotenuse.

$$\cos A = \frac{\text{side adjacent to angle } A}{\text{hypotenuse}} = \frac{b}{c} = \frac{6}{3\sqrt{5}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

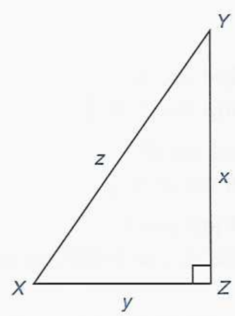
$$\cos B = \frac{\text{side adjacent to angle } B}{\text{hypotenuse}} = \frac{a}{c} = \frac{3}{3\sqrt{5}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

The tangent of an angle is the length of the side opposite the angle over the length of the side adjacent to the angle. Thus,

$$\tan A = \frac{\text{side opposite angle } A}{\text{side adjacent to angle } A} = \frac{a}{b} = \frac{3}{6} = \frac{1}{2}$$

$$\tan B = \frac{\text{side opposite angle } B}{\text{side adjacent to angle } B} = \frac{b}{a} = \frac{6}{3} = 2$$

2. Find the sine, cosine, and tangent ratios, in terms of the values x and y only, for the angle labeled X in the right triangle in the figure.



Using the Pythagorean theorem, we find the hypotenuse:

$$z^2 = x^2 + y^2$$
$$z = \sqrt{x^2 + y^2}$$

Since the sine of an acute angle in a right triangle is the length of the side opposite the angle divided by the length of the hypotenuse, then

$\sin X = \dfrac{x}{\sqrt{x^2 + y^2}}$; similarly, $\cos X = \dfrac{y}{\sqrt{x^2 + y^2}}$ and $\tan X = \dfrac{x}{y}$. ■

The reciprocal trigonometric ratios

The final three trigonometric ratios are called the **cosecant** (csc), **secant** (sec), and **cotangent** (cot) of an acute angle of a right triangle. These ratios are the reciprocals of the three ratios above.

Reciprocal trigonometric ratios

If θ represents either acute angle of a right triangle, then

$$\csc\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \cot\theta = \frac{1}{\tan\theta}$$

Note The definitions of csc θ, sec θ, and cot θ given here are equivalent to the following definitions:

$$\csc\theta = \frac{\text{hyp}}{\text{opp}}, \ \sec\theta = \frac{\text{hyp}}{\text{adj}}, \ \cot\theta = \frac{\text{adj}}{\text{opp}}$$

We could use these as the definitions for these three ratios.

It is worth stressing that the following pairs of ratios are reciprocals:

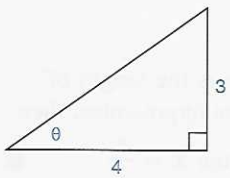
cosine and secant
sine and cosecant
tangent and cotangent

This means that if we know one ratio, we can invert it to find the other ratio in the pair.

■ ***Example 1–2 B***

1. $\sin A = \frac{2}{3}$. Find $\csc A$.
   Invert $\frac{2}{3}$, giving $\csc A = \frac{3}{2}$
2. $\cot B = 5$. Find $\tan B$.
   Invert $\frac{5}{1}$ to get $\tan B = \frac{1}{5}$
3. $\sec A = 1.6$. Find $\cos A$.
   Invert 1.6 to get $\frac{1}{1.6}$, or 0.625, so $\cos A = 0.625$. ■

■ ***Example 1–2 C***



1. Given the triangle in the figure, find the six trigonometric ratios of θ.

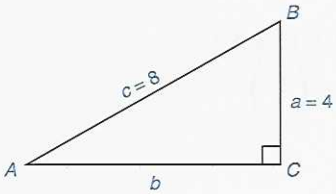
   Since we are not told the length of the hypotenuse, we calculate it, using the Pythagorean theorem:

$$3^2 + 4^2 = 9 + 16 = 25, \text{ and } \sqrt{25} = 5$$

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{3}{5}, \ \cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{4}{5}, \ \tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{3}{4},$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{3}{5}} = \frac{5}{3}, \ \sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{4}{5}} = \frac{5}{4},$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\frac{3}{4}} = \frac{4}{3}$$



2. Find $\sin B$ and $\cos B$ for the triangle in the figure.

First we find the length of side b, using the Pythagorean theorem: $a^2 + b^2 = c^2$ so $16 + b^2 = 64$, or $b^2 = 48$, so $b = \sqrt{48} = \sqrt{(16)(3)} = 4\sqrt{3}$.

$$\sin B = \frac{b}{c} = \frac{4\sqrt{3}}{8} = \frac{\sqrt{3}}{2}$$

$$\cos B = \frac{a}{c} = \frac{4}{8} = \frac{1}{2}$$

The fundamental identity of trigonometry

Recall that an identity is an equation that is true for any valid replacement of the variable. The following trigonometric identity is so important that it is often called the **fundamental identity of trigonometry.**

> **The fundamental identity of trigonometry**
>
> If θ is either acute angle in a right triangle, then
>
> $$\sin^2\theta + \cos^2\theta = 1$$
>
> ***Concept***
>
> If we square both the sine and the cosine ratios for a given angle and add these quantities, we always get 1.

Note 1. In chapter 2 we see that θ does not have to be acute.
2. The notation $\sin^2\theta$ means $(\sin \theta)^2$, and $\cos^2\theta$ means $(\cos \theta)^2$.

■ *Example 1–2 D*

1. Show that the fundamental identity of trigonometry applies to the results of part 2 of example 1–2 C.

Recall that in part 2 of example 1–2 C the angle is called B, not θ. Also, $\sin B = \dfrac{\sqrt{3}}{2}$ and $\cos B = \frac{1}{2}$.

$$\begin{aligned}\sin^2 B + \cos^2 B &= (\sin B)^2 + (\cos B)^2 \\ &= \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 \\ &= \tfrac{3}{4} + \tfrac{1}{4} \\ &= 1\end{aligned}$$

2. Prove that the fundamental identity of trigonometry applies to angle A in any right triangle ABC.

$$\begin{aligned}\sin^2 A + \cos^2 A &= (\sin A)^2 + (\cos A)^2 \\ &= \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 \\ &= \frac{a^2}{c^2} + \frac{b^2}{c^2} \\ &= \frac{a^2 + b^2}{c^2}\end{aligned}$$

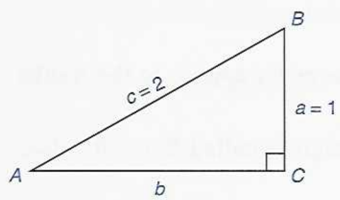
and, since $a^2 + b^2 = c^2$,

$$\begin{aligned} &= \frac{c^2}{c^2} \\ &= 1\end{aligned}$$

Finding other trigonometric ratios from a known ratio

If we know one of the trigonometric ratios of an angle, we can construct a right triangle with an angle for which that ratio is true. We can use this triangle to compute the other five trigonometric ratios.

Example 1–2 E



In right triangle ABC, $\sin A = \frac{1}{2}$. Draw a triangle in which $\sin A$ is $\frac{1}{2}$, and use this to compute the other five trigonometric ratios for angle A.

Since $\sin A$ is $\dfrac{a}{c}$, we see that $\dfrac{a}{c} = \dfrac{1}{2}$. Thus, a right triangle in which $a = 1$ and $c = 2$ would work. This is shown in the figure. We then compute b by the Pythagorean theorem,

$$\begin{aligned}a^2 + b^2 &= c^2 \\ 1^2 + b^2 &= 2^2 \\ b^2 &= 3 \\ b &= \sqrt{3}\end{aligned}$$

and then compute the other five ratios for angle A:

$$\cos A = \frac{\sqrt{3}}{2},\ \tan A = \frac{1}{\sqrt{3}} \text{ or } \frac{\sqrt{3}}{3},\ \sec A = \frac{2}{\sqrt{3}} \text{ or } \frac{2\sqrt{3}}{3},$$
$$\csc A = 2,\ \cot A = \sqrt{3}$$

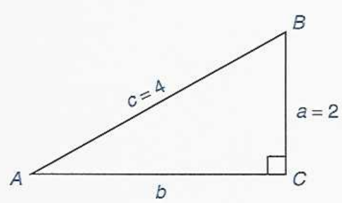


Figure 1–7

A logical question is whether any other triangles would work in example 1–2 E; the answer is yes. In fact, since $\frac{2}{4}$ reduces to $\frac{1}{2}$, we could use the triangle in figure 1–7. However, our values for the six ratios would not change.

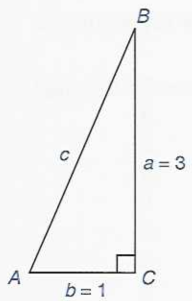
For example, we find that $b = \sqrt{12}$, or $2\sqrt{3}$; thus, $\cos A = \frac{2\sqrt{3}}{4}$, which reduces to the same value, $\frac{\sqrt{3}}{2}$. The other ratios will also reduce to the same values we already obtained.

Similarly, we could start with any fraction that is equivalent to $\frac{1}{2}$. This means that we could use an unlimited number of triangles in such problems. This is true because of the properties of similar triangles, discussed in section 1–3.

■ Example 1–2 F

1. $\cot B = 3$. Draw a right triangle for which this is true and compute the other five trigonometric ratios for B.

We know that if $\cot B = 3$, then $\tan B = \frac{1}{3}$ (see example 1–2 B) and $\tan B = \frac{\text{opp}}{\text{adj}} = \frac{b}{a}$. Thus, a triangle in which $b = 1$ and $a = 3$ would work. This is shown in the figure.



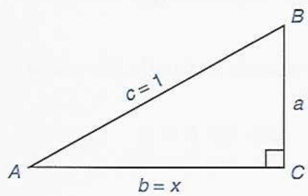
Using the Pythagorean theorem, we find c:

$$c^2 = a^2 + b^2 = 3^2 + 1^2 = 10$$
$$c = \sqrt{10}$$

With this we can now compute the four remaining ratios:

$$\sin B = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10},\ \cos B = \frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10},$$
$$\sec B = \frac{\sqrt{10}}{3},\ \csc B = \sqrt{10}$$

2. In right triangle ABC, $\cos A = x$. Draw a right triangle for which this is true and find $\tan B$ in terms of x.



If we think of x as $\frac{x}{1}$ we see that $\cos A$ in the triangle shown in the figure is x.

From the Pythagorean theorem we find a:

$$c^2 = a^2 + b^2$$
$$1^2 = a^2 + x^2$$
$$1 - x^2 = a^2$$
$$\sqrt{1 - x^2} = a$$

Now we can find $\tan B$:

$$\tan B = \frac{b}{a} = \frac{x}{\sqrt{1 - x^2}}$$

■

Mastery points

Can you

- State the definitions of the six trigonometric ratios?
- Find the six trigonometric ratios of an angle in a right triangle when you know the lengths of the sides?
- State and use the fundamental identity of trigonometry?
- Find the remaining five ratios for an angle if given one of the six ratios for an angle in a right triangle?

Exercise 1–2

1. Draw a right triangle, label it in the standard way (using A, B, C, a, b, c), and use it to define the six trigonometric ratios for both acute angles.

In the following problems you are given parts of right triangle ABC; use this information to compute the six trigonometric ratios for the angle specified.

	a	b	c	Find ratios for this angle		a	b	c	Find ratios for this angle
2.	3	4		A	**3.**	3	4		B
4.	5		13	A	**5.**	1	3		B
6.	4	$\sqrt{10}$		A	**7.**	5	$\sqrt{7}$		B
8.		2	$\sqrt{13}$	A	**9.**	2		$\sqrt{5}$	B
10.	4		7	A	**11.**	12	13		B
12.	5	12		A	**13.**	6		10	A
14.	10		15	A	**15.**	$\sqrt{3}$	4		A
16.	x	y		A	**17.**	x		z	B
18.	1	1		A	**19.**	1		2	B
20.		5	8	A	**21.**	9	5		B

22. You will learn in section 1–3 that if the sine ratio for an acute angle is more than 0.5, then the angle is larger than 30°. In a right triangle, $a = 3$ and $c = 5$; is angle B more or less than 30°?

In the following problems you are given one of the trigonometric ratios for an angle. Use this to sketch a triangle for which the ratio is true, and then use this triangle to find the other five trigonometric ratios for that angle.

23. $\sin A = \frac{4}{5}$
24. $\cos B = \frac{1}{4}$
25. $\cos A = 0.5$
26. $\tan B = 4$
27. $\sec A = 3$
28. $\cot B = 0.2$
29. $\sin A = \frac{5}{13}$
30. $\csc B = 1.6$
31. $\cos A = 0.9$

Solve the following problems.

32. Show that the fundamental identity of trigonometry applies to the results of problems 27 and 29.

33. Show that the fundamental identity of trigonometry applies to angle B in any right triangle ABC.

34. In right triangle ABC, $\sin A = x$. Sketch a triangle for which this is true and use it to find $\sin B$ in terms of the variable x. $\left(\text{Hint: } x = \dfrac{x}{1}.\right)$

35. In right triangle ABC, $\tan A = x$. Sketch a triangle for which this is true and use it to find $\sin B$ in terms of the variable x. (See the hint for problem 34.)

36. In right triangle ABC, $\sec A = x$. Sketch a triangle for which this is true and use it to find $\sec B$ in terms of the variable x.

37. In right triangle ABC, $\tan B = x$. Sketch a triangle for which this is true and use it to find $\tan A$ in terms of the variable x.

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1–3 Angle measure and the values of the trigonometric ratios

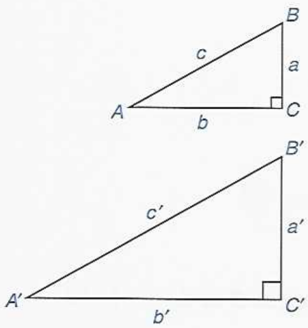


Figure 1–8

Trigonometric ratios for equal angles are equal

It is possible to have angles of the same measure in different right triangles. Angles A and A' in figure 1–8 are examples. The trigonometric ratios will have the same value for these angles. This is because these triangles are *similar*—that is, they have the same shape, but perhaps different sizes. It is a theorem of geometry that corresponding ratios in similar figures are equal. Thus, in figure 1–8, $\frac{a}{c} = \frac{a'}{c'}$, so $\sin A = \sin A'$. For the same reasons, the other five trigonometric ratios are also equal.

Note Read A' as "A-prime," B' as "B-prime," etc.

These facts mean that the values of the trigonometric ratios for an acute angle do not depend upon the particular right triangle in which it appears. *For an acute angle with a given measure, the values of the trigonometric ratios will always be the same.*

Values for angles of measure 30°, 45°, and 60°

We now find the trigonometric ratios for some special angles—30°, 45°, and 60°. Figure 1–9 shows an equilateral triangle, which is a triangle in which all sides have equal length. We label this length c.

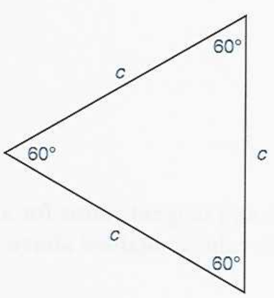


Figure 1–9

We construct the line AC, as shown in figure 1–10, which forms a right triangle with acute angles $A = 30°$ and $B = 60°$. The length a is half of c, so $a = \frac{c}{2}$. We find b next.

$$b^2 = c^2 - a^2 = c^2 - \left(\frac{c}{2}\right)^2 = \frac{4c^2}{4} - \frac{c^2}{4}$$

$$b^2 = \frac{3c^2}{4}$$

$$b = \frac{\sqrt{3}}{2}c$$

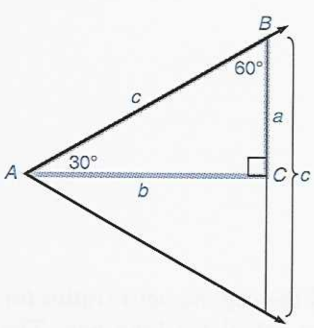


Figure 1–10

Angle A is a 30° angle, so we will write sin 30° to mean the value of the sine ratio associated with an angle of measure 30°. Thus,

$$\sin 30^\circ = \frac{a}{c} = \frac{\frac{c}{2}}{c} = \frac{1}{2}$$

$$\cos 30^\circ = \frac{b}{c} = \frac{\frac{\sqrt{3}}{2}c}{c} = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{a}{b} = \frac{\frac{c}{2}}{\frac{\sqrt{3}}{2}c} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

The values for a 60° angle can be obtained from angle B in figure 1–10.

$$\sin 60^\circ = \frac{b}{c} = \frac{\frac{\sqrt{3}}{2}c}{c} = \frac{\sqrt{3}}{2}$$

$$\cos 60^\circ = \frac{a}{c} = \frac{\frac{c}{2}}{c} = \frac{1}{2}$$

$$\tan 60^\circ = \frac{b}{a} = \frac{\frac{\sqrt{3}}{2}c}{\frac{c}{2}} = \sqrt{3}$$

In the exercises we will compute the sine, cosine and tangent ratios for a 45° angle; these are shown in table 1–1, along with the values obtained above.

	Sine	**Cosine**	**Tangent**
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

Table 1–1

General values

It is actually impossible to find the exact values of the trigonometric ratios for most angles. Tables of approximate values were calculated long ago. The earliest known table of trigonometric values, for the equivalent of the sine ratio, was created by Hipparchus of Nicaea about 150 B.C. In the second century A.D. Ptolemy constructed a table of values of the sine ratio for acute angles

in increments of one-quarter degree. Today we use calculators to approximate these values. When using a calculator it is important that the calculator be in **degree mode.** This means that the calculator is expecting the measure of the angle in decimal degrees. *Check your calculator's manual to make sure it is in degree mode.* This is usually done with a key marked [DRG] or simply [DEG]. "DRG" means degrees, radians, grads. We discuss radian measure in a later section. Grads, or grades, is the metric measure for an angle. There are 100 grads in a right angle. We will not use this measure in this text.

To select degree mode on the Texas Instruments TI-81 it is necessary to select "Deg" under the [MODE] feature. To do this, select [MODE], darken in the "Deg" mode indicator (use the four cursor-moving "arrow" keys) and select [ENTER].

■ Example 1–3 A

Find each value rounded to four decimal places.

1. $\sin 34.51°$ — 34.51 [sin] Display $\boxed{0.5665500655}$

$\sin 34.51° \approx 0.5666$ — [TI-81] [SIN] 34.51 [ENTER]

2. $\tan 84.6°$ — 84.6 [tan] Display $\boxed{10.57889499}$

$\tan 84.6° \approx 10.5789$ — [TI-81] [TAN] 84.6 [ENTER]

3. $\sec 33.5°$

Since there is no secant key on a calculator we use the fact that $\sec 33.5° = \dfrac{1}{\cos 33.5°}$. Compute $\cos 33.5°$ and divide it into one; the [$1/x$] key is designed for this type of situation.

33.5 [cos] [$1/x$] Display $\boxed{1.199204943}$

[TI-81] [(] [COS] 33.5 [)] [x^{-1}] [ENTER]

Note that on the TI-81 the [$1/x$] key is the [x^{-1}] key.

$\sec 33.5° \approx 1.1992$

4. $\cot 87.23°$ — $\cot 87.23° = \dfrac{1}{\tan 87.23°}$, so use

87.23 [tan] [$1/x$]

Display $\boxed{0.04838332158}$

[TI-81] [(] [TAN] 87.23 [)] [x^{-1}] [ENTER]

$\cot 87.23° \approx 0.0484$

5. cos 13°43′
Recall that angles in the DMS system must be converted to decimal degrees. We show the calculation with and without special calculator keys. (See also example 1–1 A.)

No special keys: 13 [+] 43 [÷] 60 [=] [cos]
Display [0.9714801855]

Calculator A: 13 [° ′ ″] 43 [° ′ ″] [cos]

Calculator B: 13.43 [→H] [cos]

[TI-81] [COS] [(] 13 [+] 43 [÷] 60 [)] [ENTER]

cos 13°43′ ≈ 0.9715

Finding an angle from a known trigonometric ratio

It is important to be able to reverse the operations discussed above. For example, if θ is an acute angle and sin θ = $\frac{1}{2}$, what is θ? We can see from table 1–1 that θ must be 30°. The calculator is programmed to solve this problem. This is done with the **inverse trigonometric ratios** called the inverse sine ($\sin^{-1}$), inverse cosine ($\cos^{-1}$) and inverse tangent ($\tan^{-1}$) ratios. The superscript −1 does *not* indicate a reciprocal value in the way that, say, $2^{-1} = \frac{1}{2}$. We will study these ratios in more detail later. For now we illustrate how to find the acute angle whose sine, cosine, or tangent value is known.

For this, most calculators use the appropriate key (sin, cos, tan), prefixed by another key such as [SHIFT], [2nd], [INV] or [ARC]. The appropriate function is generally shown above the key itself. The *result is always an angle in decimal degrees* (when the calculator is in degree mode). We will show the necessary two keystrokes as one.

$\sin^{-1}$	$\cos^{-1}$	$\tan^{-1}$
[sin]	[cos]	[tan]

■ Example 1–3 B

Find θ in the following problems using the calculator. Assume θ is an acute angle. Round the answer to the nearest 0.01°.

1. cos θ = 0.4602
We need to calculate $\cos^{-1}0.4602$.

0.4602 [$\cos^{-1}$] Display [62.59998611]

[TI-81] [COS^{-1}] .4602 [ENTER]

θ ≈ 62.60°

2. tan θ = 1.2231
Calculate $\tan^{-1}1.2231$.

1.2231 [$\tan^{-1}$] Display [50.73075144]

[TI-81] [TAN^{-1}] 1.2231 [ENTER]

θ ≈ 50.73°

3. $\csc\theta = 1.0551$

We use the fact that if $\csc\theta = 1.0551$, then $\sin\theta = \dfrac{1}{1.0551}$. Thus we compute $\sin^{-1}\left(\dfrac{1}{1.0551}\right)$. Use the $\boxed{1/x}$ key. (On the TI-81 the $\boxed{1/x}$ key is the $\boxed{x^{-1}}$ key.)

1.0551 $\boxed{1/x}$ $\boxed{\sin^{-1}}$ Display $\boxed{71.40162609}$

$\boxed{\text{TI-81}}$ $\boxed{\text{SIN}^{-1}}$ 1.0551 $\boxed{x^{-1}}$ $\boxed{\text{ENTER}}$

$\theta \approx 71.40°$

Solving right triangles

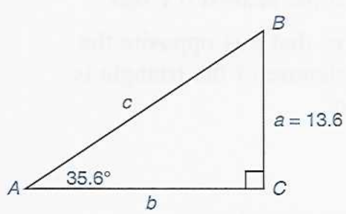
One application of trigonometry that occurs in many situations is *solving right triangles*—this means *discovering the lengths of all sides and the measures of all angles* of the triangle. We will round the values we compute to the same number of decimal places as the given data.

These types of situations fall into two categories, ones in which we know one side and one acute angle and others in which we know two sides and no angles. Each category is illustrated in example 1–3 C.

■ *Example 1–3 C*

Solve the following right triangles.

1. $A = 35.6°$, $a = 13.6$ (one side, one angle)



To solve this triangle we need to find the lengths of sides b and c and the measure of angle B. Since angle C is always 90°, angles A and B total 90°. Thus, angle B is $90° - 35.6° = 54.4°$. We now note that $\sin A = \dfrac{a}{c}$, so that

$$\sin 35.6° = \frac{13.6}{c}$$

$$c \sin 35.6° = 13.6 \quad \text{Multiply each member by } c$$

$$c = \frac{13.6}{\sin 35.6°} \quad \text{Divide each member by } \sin 35.6°$$

$$c \approx 23.4 \quad \boxed{\text{CS 1}}$$

Now we find b by noting that $\tan A = \dfrac{a}{b}$.

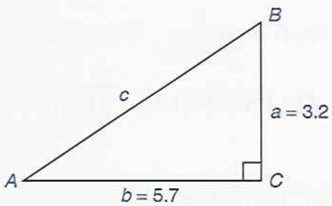
$$\tan 35.6° = \frac{13.6}{b}$$

$$b \tan 35.6° = 13.6 \quad \text{Multiply each member by } b$$

$$b = \frac{13.6}{\tan 35.6°} \quad \text{Divide each member by } \tan 35.6°$$

$$b \approx 19.0 \quad \boxed{\text{CS 2}}$$

Since we know the lengths of all sides and all angles we have solved the triangle. To summarize, $a = 13.6$, $b \approx 19.0$, $c \approx 23.4$, $A = 35.6°$, $B = 54.4°$, $C = 90°$.



2. $a = 3.2$, $b = 5.7$ (two sides)
We can find the length of side c by the Pythagorean theorem.

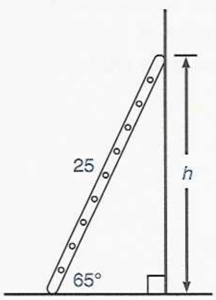
$$c^2 = a^2 + b^2$$
$$c^2 = 3.2^2 + 5.7^2$$
$$c^2 = 42.73$$
$$c = \sqrt{42.73}$$
$$c \approx 6.5$$

We can find angle A by noting that $\tan A = \dfrac{a}{b}$.

$$\tan A = \frac{3.2}{5.7}$$
$$A = \tan^{-1}\left(\frac{3.2}{5.7}\right) \qquad \boxed{\text{CS 3}}$$
$$A \approx 29.3°$$

We know that the sum of the measures of A and B is 90°, so $B = 90° - 29.3° = 60.7°$. This completes the process of finding the lengths of all sides and measures of all angles. Thus, $a = 3.2$, $b = 5.7$, $c \approx 6.5$, $A \approx 29.3°$, $B \approx 60.7°$, $C = 90°$.

3. A tag on a 25-foot ladder states that, for safety reasons, the angle that the ladder makes with the ground should not exceed 65°. How high can the ladder reach without exceeding this angle, to the nearest 0.1 feet?



We need to find h in the figure. If we observe that h is opposite the known angle and that the length of the hypotenuse of the triangle is known, we see that we can use the sine ratio.

$$\sin 65° = \frac{h}{25}$$
$$25 \sin 65° = h \qquad \text{Multiply each member by 25}$$
$$22.7 \approx h \qquad \boxed{\text{CS 4}}$$

The ladder can reach a height of approximately 22.7 feet without exceeding a 65° angle with the ground.

Calculator steps

1. Ⓐ	13.6 [÷] 35.6 [sin] [=]	Display	[23.36276130]
Ⓟ	13.6 [ENTER] 35.6 [sin] [÷]		
[TI-81]	13.6 [÷] [SIN] 35.6 [ENTER]		
2. Ⓐ	13.6 [÷] 35.6 [tan] [=]	Display	[18.99627899]
Ⓟ	13.6 [ENTER] 35.6 [tan] [÷]		
[TI-81]	13.6 [÷] [TAN] 35.6 [ENTER]		

3. Ⓐ 3.2 $\boxed{\div}$ 5.7 $\boxed{=}$ $\boxed{\tan^{-1}}$ Display $\boxed{29.31000707}$

Ⓟ 3.2 $\boxed{\text{ENTER}}$ 5.7 $\boxed{\div}$ $\boxed{\tan^{-1}}$

$\boxed{\text{TI-81}}$ $\boxed{\text{TAN}^{-1}}$ $\boxed{(}$ 3.2 $\boxed{\div}$ 5.7 $\boxed{)}$ $\boxed{\text{ENTER}}$

4. Ⓐ 25 $\boxed{\times}$ 65 $\boxed{\sin}$ $\boxed{=}$ Display $\boxed{22.65769468}$

Ⓟ 25 $\boxed{\text{ENTER}}$ 65 $\boxed{\sin}$ $\boxed{\times}$

$\boxed{\text{TI-81}}$ 25 $\boxed{\times}$ $\boxed{\text{SIN}}$ 65 $\boxed{\text{ENTER}}$

Mastery points

Can you

- Compute the value of the trigonometric ratios for a given acute angle, using a calculator?
- Compute the value of an acute angle, given the value of a trigonometric ratio, using a calculator?
- Solve a right triangle when given one side and one acute angle?
- Solve a right triangle when given two sides?

Exercise 1–3

Use a calculator to find four-decimal-place approximations for the following.

1. sin 31.28°
2. cos 85.23°
3. tan 11.95°
4. sec 40.08°
5. cot 28.87°
6. csc 5.15°
7. sin 40.28°
8. tan 76.23°
9. sec 66.47°
10. sin 35.56°
11. sin 78.33°
12. cos 17.45°
13. sin 78°33′
14. cos 17°45′
15. cos 85°28′
16. tan 40°41′
17. tan 35°8′
18. cos 23°24′
19. cos 56°24′
20. cot 13°3′
21. sin 48°8′
22. tan 33°38′
23. sec 86°22′

24. A surveyor needs to compute R in the following formula as part of finding the area of the segment of a circle: $R = \dfrac{LC}{2 \sin I}$. Find R to three decimal places if $LC = 425.0$ feet and $I = 13.2°$.

25. Compute R using the formula of the previous problem if $LC = 611.1$ meters and $I = 18°20'$. Round the answer to two decimal places.

26. In the mathematical modeling of an aerodynamics problem the following equation arises:
$$y = x \cos A \cos B - x^2 \cos A \sin B - x^3 \sin A$$
Compute y to two decimal places if $x = 2.5$, $A = 31°$, and $B = 17°$.

27. Compute y to two decimal places using the formula of problem 26 if $x = 1.2$, $A = 10°$, and $B = 15°$.

28. The average power in an AC circuit is given by the formula $P = VI \cos \theta$. Compute P (in watts) if $V = 120$ volts, $I = 2.3$ amperes, and $\theta = 45°$, to the nearest 0.1 watt.

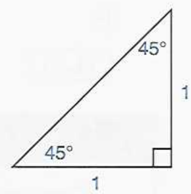
29. Compute P using the formula of problem 28 if $V = 42$ volts, $I = 25$ amperes, and $\theta = 45°$, to the nearest 0.1 watt.

30. A formula that relates the distance across the flats of a piece of hexagonal stock in relation to the distance across the corners is $f = 2r \cos \theta$. A machinist needs to compute f for a piece of stock in which $r = 28$ millimeters (mm) and $\theta = 30°$. Compute f to the nearest 0.1 mm.

31. Using the formula of problem 30 find r if $f = 21.4$ inches and $\theta = 25°$.

32. Using the formula of problem 30 find θ to the nearest 0.1 if $f = 36.8$ millimeters and $r = 24.0$.

33. Find the exact values for the sine, cosine, and tangent ratios for an angle of measure 45° by proceeding in the following manner. Draw an *isosceles* right triangle—a right triangle in which the two legs have the same length. Label this length one. Observe that the two acute angles must be 45°. Now find the length of the hypotenuse and use the definitions of the trigonometric ratios to find the desired values.



Find the unknown acute angle θ to the nearest 0.01°.

34. $\sin \theta = 0.3746$ **35.** $\sin \theta = 0.8007$ **36.** $\cos \theta = 0.1028$
37. $\tan \theta = 1.8807$ **38.** $\sin \theta = 0.9484$ **39.** $\cos \theta = 0.8515$
40. $\tan \theta = 1.0014$ **41.** $\sin \theta = \dfrac{35.9}{68.3}$ **42.** $\cos \theta = \dfrac{8.25}{12.5}$
43. $\tan \theta = 2$ **44.** $\csc \theta = 1.1243$ **45.** $\sec \theta = 4.8097$
46. $\cot \theta = 2.5$ **47.** $\sec \theta = \dfrac{6.45}{2.35}$ **48.** $\csc \theta = \sqrt{10.8}$

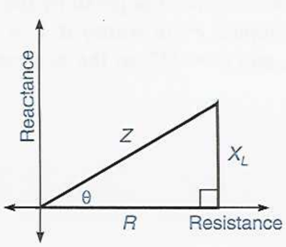
In the following problems you are given one side and one angle of a right triangle. Solve the triangle. Round all answers to the same number of decimal places as the data.

49. $a = 15.2, B = 38.3°$ **50.** $a = 12.6, B = 17.9°$ **51.** $a = 11.1, A = 13.7°$
52. $a = 5.25, A = 70.3°$ **53.** $b = 0.672, A = 29.4°$ **54.** $b = 15.2, A = 81.3°$
55. $b = 21.8, B = 78.0°$ **56.** $b = 2.14, B = 50.4°$ **57.** $c = 10.0, A = 15.0°$
58. $c = 3.45, A = 46.2°$ **59.** $c = 122, B = 65.5°$ **60.** $c = 31.5, B = 62.0°$

In the following problems you are given two sides of a right triangle. Solve the triangle. Round all lengths to the same number of decimal places as the data and all angles to the nearest 0.1°.

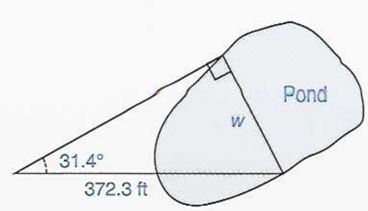
61. $a = 13.1, b = 15.6$ **62.** $a = 5.67, b = 8.91$ **63.** $a = 0.22, b = 1.34$
64. $a = 2.82, b = 1.09$ **65.** $a = 17.8, c = 25.2$ **66.** $a = 311, c = 561$
67. $b = 51.3, c = 111.0$ **68.** $b = 4.55, c = 5.66$ **69.** $a = 12.0, c = 13.0$
70. $a = 33.1, c = 41.0$ **71.** $b = 84.0, c = 90.1$

72. The figure illustrates an impedance diagram used in electronics theory. If Z (impedance) = 10.35 ohms and X_L (inductive reactance) = 4.24 ohms, find θ (phase angle) to the nearest degree and R (resistance) to the nearest 0.01 ohm.

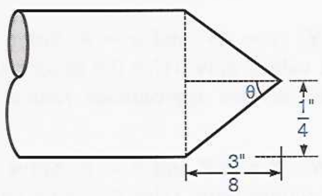


73. Use the impedance diagram of problem 72 to find Z if $\theta = 24.2°$ and $X_L = 22.6$ ohms.

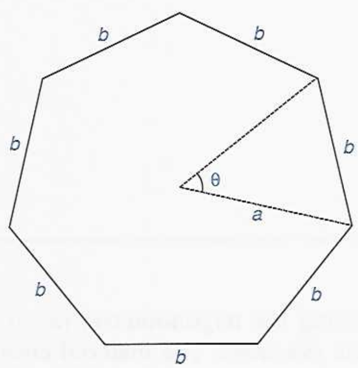
74. The diagram illustrates the measurements a surveyor made to find the width w of a pond; compute the width to the nearest foot.



75. The diagram illustrates the tip of a threading tool; find angle θ to the nearest degree.

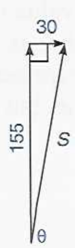


76. The diagram illustrates a piece of wood that is being mass produced to form the bottom of a planter. Find dimension a in the figure to the nearest 0.01 inch if $b = 8\frac{1}{4}$ inches. Note: angle θ is $\frac{360°}{7}$.

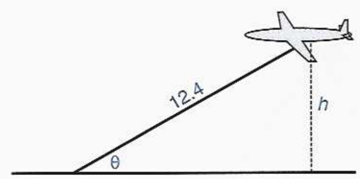


77. A formula found in electronics is $E = \frac{P}{I \cos \theta}$, where E is voltage, P is power, I is current, and θ is phase angle. Find E (in volts) if $P = 45.0$ watts, $I = 2.5$ amperes, and $\theta = 15°$. Round the answer to the nearest 0.1 volt.

78. The diagram illustrates the wind triangle problem in air navigation. A plane has an airspeed of 155 mph and heading of due north. It is flying in a wind from the west with a speed of 30 mph. Find the ground speed S and the ground direction θ, each to the nearest unit.

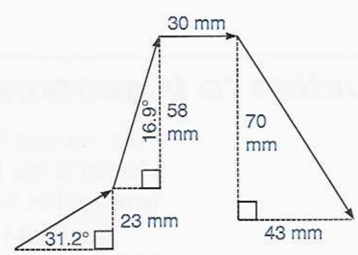


79. An **angle of elevation** is an angle formed by one horizontal ray and another ray that is above the horizontal. Angle θ in the diagram is the angle of elevation to an aircraft that radar shows has a slant distance of 12.4 miles from the radar site. If θ is 30.1°, find the elevation h of the aircraft, to the nearest 100 feet. Remember that 1 mile = 5,280 feet.

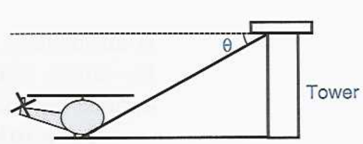


80. If it is known that an aircraft is flying at 28,500 feet and the angle of elevation of a radar beam tracking the aircraft is 8.2°, what is the slant distance d from the radar to the aircraft, to the nearest 100 feet?

81. The diagram illustrates the path of a laser beam on an optics table. Compute the total distance traveled by the beam to the nearest millimeter.

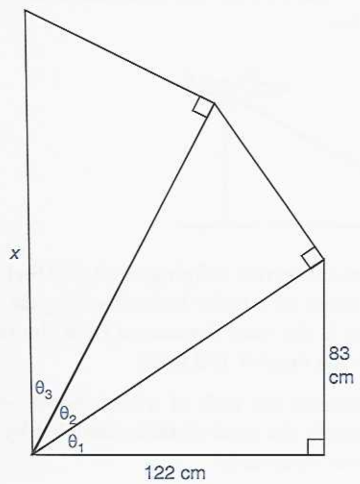


82. An **angle of depression** is an angle formed by one horizontal ray and another ray that is below the horizontal. Angle θ in the figure is the angle of depression formed by the line of sight of an observer in an airport control tower looking at a helicopter on the ground. If θ is 17.2° and the tower is 257 feet high, how far is the aircraft from the base of the tower, to the nearest foot.



83. If an aircraft is 1.23 miles from the foot of the tower in problem 82, what is θ, to the nearest 0.1°? (Remember, 1 mile = 5,280 ft.)

84. The diagram is a top view of a portion of a spiral staircase that an architect has designed. If $\theta_1 = \theta_2 = \theta_3$, find the length x to one decimal place. (Caution: Carry out your calculations to as many digits as practical to avoid an accumulation of errors.)



85. If the architect of problem 84 revises the plans so that $\theta_2 = \theta_1 + 5°$ and $\theta_3 = \theta_2 + 5°$, find x to one decimal place.

86. In right triangle ABC, $A = 45°$ and $b = 4$. Solve this triangle using exact values only. (Use the exact values for sin 45° etc., and do not approximate radicals as decimals.)

87. In right triangle ABC, $B = 60°$ and $b = 8$. Solve this triangle using exact values only. (Use the exact values for sin 60° etc., and do not approximate radicals as decimals.)

1–4 Introduction to trigonometric equations

This section introduces equations involving the trigonometric ratios. In chapter 2 we learn about the trigonometric *functions.* The material covered here applies to these functions as well.

Equations can be categorized as identities and conditional equations. An **identity** is an equation that is true for every allowed value of its variable (or variables). For example,

$$2(x + 3) = 2x + 6$$

is an identity, since the left member and right member of the equation represent the same value, regardless of the value of x. Similarly

$$\frac{3x^2}{3x} = x$$

is an identity; the left member equals the right member for every value of x for which both members are defined. Observe that the left member is not defined for the value 0, so the identity is true for all real values *except* zero.

A **conditional equation** is an equation that is true only for some, but not all, values that may replace the variable. For example,

$$6x = 12$$

is true if and only if x is replaced by 2, and

$$x^2 = 9$$

is true if and only if x is replaced by 3 or -3.

Identities

We have seen the reciprocal ratio identities

$$\csc\theta = \frac{1}{\sin\theta}, \sec\theta = \frac{1}{\cos\theta}, \cot\theta = \frac{1}{\tan\theta}$$

Similarly,

$$\sin\theta = \frac{1}{\csc\theta}, \cos\theta = \frac{1}{\sec\theta}, \tan\theta = \frac{1}{\cot\theta}$$

are identities.

Knowing these identities permits us to simplify certain trigonometric expressions.

■ *Example 1–4 A*

Simplify each trigonometric expression.

1. $\csc\theta \sin\theta$

$$\csc\theta\sin\theta$$

$$\frac{1}{\sin\theta}\cdot\sin\theta \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$1$$

Thus, $\csc\theta\sin\theta = 1$.

2. $\dfrac{1-\csc\theta}{\csc\theta}$

$$\frac{1-\csc\theta}{\csc\theta}$$

$$\frac{1}{\csc\theta} - \frac{\csc\theta}{\csc\theta} \qquad \frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}$$

$$\sin\theta - 1 \qquad \sin\theta = \frac{1}{\csc\theta}$$

Thus, $\dfrac{1-\csc\theta}{\csc\theta} = \sin\theta - 1$. The right member is considered simpler because it is not a rational expression. ■

Two more useful identities are

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \cot\theta = \frac{\cos\theta}{\sin\theta}$$

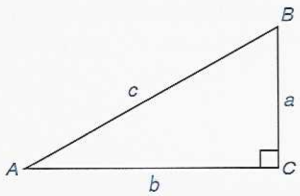


Figure 1–11

To see why the first is true consider angle A in figure 1–11. Note that

$\tan A = \dfrac{a}{b}$, and $\dfrac{\sin A}{\cos A} = \dfrac{\frac{a}{c}}{\frac{b}{c}} = \dfrac{a}{c}\cdot\dfrac{c}{b} = \dfrac{a}{b}$ also. It is left as an exercise to show that the same is true for angle B and that the second identity is true.

■ ***Example 1–4 B***

Simplify each expression.

1. $\cot\alpha(\sin\alpha - \tan\alpha)$

$\cot\alpha(\sin\alpha - \tan\alpha)$

$\cot\alpha\sin\alpha - \cot\alpha\tan\alpha$ — $a(b - c) = ab - ac$

$\dfrac{\cos\alpha}{\sin\alpha}\sin\alpha - \dfrac{1}{\tan\alpha}\tan\alpha$ — $\cot\alpha = \dfrac{\cos\alpha}{\sin\alpha}$; $\cot\alpha = \dfrac{1}{\tan\alpha}$

$\cos\alpha - 1$

Note We replaced $\cot\alpha$ by $\dfrac{\cos\alpha}{\sin\alpha}$ in one term and by $\dfrac{1}{\tan\alpha}$ in another term. We use whichever identity better suits the rest of the term.

Thus, $\cot\alpha(\sin\alpha - \tan\alpha) = \cos\alpha - 1$.

2. $\sec\theta(\cos\theta - \cot\theta)$

$\sec\theta(\cos\theta - \cot\theta)$

$\sec\theta\cos\theta - \sec\theta\cot\theta$

$\dfrac{1}{\cos\theta}\cos\theta - \dfrac{1}{\cos\theta}\cdot\dfrac{\cos\theta}{\sin\theta}$

$1 - \dfrac{1}{\sin\theta}$

$1 - \csc\theta$

Thus $\sec\theta(\cos\theta - \cot\theta) = 1 - \csc\theta$. ■

We also use the fundamental identity of trigonometry (section 1–2):

$$\sin^2\theta + \cos^2\theta = 1$$

■ ***Example 1–4 C***

1. Simplify the expression $\dfrac{1}{\sec^2\theta} + \dfrac{1}{\csc^2\theta}$.

$\dfrac{1}{\sec^2\theta} + \dfrac{1}{\csc^2\theta}$

$\dfrac{1}{(\sec\theta)^2} + \dfrac{1}{(\csc\theta)^2}$ — $\sec^2\theta$ means $(\sec\theta)^2$, $\csc^2\theta$ means $(\csc\theta)^2$

$\left(\dfrac{1}{\sec\theta}\right)^2 + \left(\dfrac{1}{\csc\theta}\right)^2$ — $\dfrac{1}{x^2} = \left(\dfrac{1}{x}\right)^2$

$(\cos\theta)^2 + (\sin\theta)^2$ — $\dfrac{1}{\sec\theta} = \cos\theta$, $\dfrac{1}{\csc\theta} = \sin\theta$

1 — The fundamental identity of trigonometry

Thus, $\dfrac{1}{\sec^2\theta} + \dfrac{1}{\csc^2\theta} = 1$.

2. Simplify the expression $1 - \sin^2\beta$.

$1 - \sin^2\beta$

$(\sin^2\beta + \cos^2\beta) - \sin^2\beta$ — $1 = \sin^2\beta + \cos^2\beta$

$\cos^2\beta$

Thus, $1 - \sin^2\beta = \cos^2\beta$.

3. Verify the fundamental identity of trigonometry for $\theta = 60°$.

$$\sin^2 60° + \cos^2 60° = \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 \quad \sin 60° = \frac{\sqrt{3}}{2},\ \cos 60° = \frac{1}{2}$$
$$= \tfrac{3}{4} + \tfrac{1}{4}$$
$$= 1$$

Conditional trigonometric equations

Section 1–3 showed that if we have an equation like $\tan\theta = 4$, then one value of θ is $\tan^{-1}4 \approx 76°$. This is an example of a simple conditional trigonometric equation. Solutions to such equations rely on the inverse sine, cosine, and tangent functions as illustrated in section 1–3.

■ *Example 1–4 D*

Solve the following conditional equations to the nearest 0.1°.

1. $2 \sin x = 1$

$\sin x = \frac{1}{2}$ — Divide both members by 2

$x = 30°$ — Table 1–1, section 1–3

2. $5 \sin x = 3$

$\sin x = \frac{3}{5}$ — Divide both members by 5

$x = \sin^{-1}\frac{3}{5}$

$x \approx 36.9°$ — CS 1

3. $\sin 5x = 0.8$

$5x = \sin^{-1}0.8$

$x = \dfrac{\sin^{-1}0.8}{5}$ or $\frac{1}{5}\sin^{-1}0.8$

$x \approx 10.6°$ — CS 2

4. $4 \cos 3\alpha = 3$

$\cos 3\alpha = \frac{3}{4}$ — Divide both members by 4

$3\alpha = \cos^{-1}\frac{3}{4}$

$\alpha = \dfrac{\cos^{-1}\frac{3}{4}}{3}$ or $\frac{1}{3}\cos^{-1}\frac{3}{4}$ — Divide both members by 3

$\alpha \approx 13.8°$ — CS 3

Observe in example 1–4 D when it is proper to divide, and when it is not. An expression like

$$10 \sin x$$

indicates the product of 10 and $\sin x$. Thus, for example, the expression

$$\frac{10 \sin x}{5} = \frac{10}{5}\sin x = 2 \sin x$$

However,

$$\frac{\sin 10x}{5}$$

would not reduce. This is because the 5 is not dividing a product. The expression "sin 10x" does not represent multiplication.

An expression like $\sin \frac{10x}{5}$ can be simplified to $\sin 2x$, since the 5 is dividing the product $10x$.

Calculator steps

1. Ⓐ 3 $\boxed{\div}$ 5 $\boxed{=}$ $\boxed{\sin^{-1}}$

   Ⓟ 3 $\boxed{\text{ENTER}}$ 5 $\boxed{\div}$ $\boxed{\sin^{-1}}$

   $\boxed{\text{TI-81}}$ $\boxed{\text{SIN}^{-1}}$ $\boxed{(}$ 3 $\boxed{\div}$ 5 $\boxed{)}$ $\boxed{\text{ENTER}}$

2. Ⓐ .8 $\boxed{\sin^{-1}}$ $\boxed{\div}$ 5 $\boxed{=}$

   Ⓟ .8 $\boxed{\sin^{-1}}$ 5 $\boxed{\div}$

   $\boxed{\text{TI-81}}$ $\boxed{\text{SIN}^{-1}}$ .8 $\boxed{\div}$ 5 $\boxed{\text{ENTER}}$

3. Ⓐ 3 $\boxed{\div}$ 4 $\boxed{=}$ $\boxed{\cos^{-1}}$ $\boxed{\div}$ 3 $\boxed{=}$

   Ⓟ 3 $\boxed{\text{ENTER}}$ 4 $\boxed{\div}$ $\boxed{\cos^{-1}}$ 3 $\boxed{\div}$

   $\boxed{\text{TI-81}}$ $\boxed{\text{COS}^{-1}}$ $\boxed{(}$ 3 $\boxed{\div}$ 4 $\boxed{)}$ $\boxed{\div}$ 3 $\boxed{\text{ENTER}}$

Mastery points

Can you

- Simplify simple trigonometric expressions?
- Solve simple equations involving the trigonometric ratios?

Exercise 1–4

Simplify the following trigonometric expressions.

1. $\tan\theta \cot\theta$
2. $\sec\theta \cos\theta$
3. $\cos\theta(1 - \sec\theta)$
4. $\cot\alpha(\tan\alpha + \sin\alpha)$
5. $\sec\theta(\cot\theta + \cos\theta - 1)$
6. $\frac{\cos\theta - 1}{\sin\theta}$
7. $\frac{\cos\alpha - \sin\alpha}{\cos\alpha}$
8. $\boxed{8.}$ $\frac{\sin\theta + \cos\theta - 2}{\cos\theta}$
9. $1 - \cos^2\theta$
10. $\cos\theta\cos\theta + \sin^2\theta$
11. $\cos\beta(\sec\beta - \cos\beta)$
12. $-\sin\theta(\sin\theta - \csc\theta)$
13. $(\cos\theta + \sin\theta)(\cos\theta - \sin\theta) + 2\sin^2\theta$
14. Verify by computation that the fundamental identity is true when $\theta = 30°$.
15. Using approximate values check the fundamental identity when
    a. $\theta = 16°50'$ **b.** $\theta = 50°$
16. Use the two identities $\cot\theta = \frac{1}{\tan\theta}$ and $\tan\theta = \frac{\sin\theta}{\cos\theta}$ to show that $\cot\theta = \frac{\cos\theta}{\sin\theta}$.
17. Use approximate values to show that $\tan\theta = \frac{\sin\theta}{\cos\theta}$ when $\theta = 32°40'$.
18. Show that $\tan\theta = \frac{\sin\theta}{\cos\theta}$ when $\theta = 60°$ (use values from table 1–1).

Use identities to show that the left side of each equation can be simplified to become the right side.

19. $\tan x(\cot x + \csc x) = 1 + \sec x$

20. $\csc \alpha(\cos \alpha - \sin \alpha) = \cot \alpha - 1$

21. $\sin \beta(\cot \beta - \csc \beta + \sin \beta) = \cos \beta - \cos^2\beta$

22. $\dfrac{\sin x - \cos x}{\sin x} = 1 - \cot x$

23. $\cos \alpha(\csc \alpha + \sec \alpha) = \cot \alpha + 1$

24. $\tan \beta(\cot \beta - \cos \beta) = 1 - \sin \beta$

Solve the following conditional equations to the nearest 0.1°.

25. $2 \cos x = 1$

$\boxed{26.}$ $\sqrt{3} \tan x = 1$

27. $2 \sin x = \sqrt{3}$

28. $\sqrt{2} \cos x = 1$

29. $5 \sin x = 1$

30. $3 \sin x = 2$

31. $2 \tan x = 9$

32. $4 \cot x = 3$

33. $\dfrac{\sin x}{3} = \dfrac{2}{11}$

34. $\dfrac{\csc x}{3} = 2$

35. $\sin 3x = \frac{1}{2}$

36. $\cos 2x = \frac{1}{2}$

37. $\tan 2x = \sqrt{3}$

38. $\sin 2x = 0.8$

39. $\csc 3x = 3$

40. $4 \sin 2x = 3$

41. $2 \cos 4x = 1$

$\boxed{42.}$ $3 \sin 2x = 0.75$

43. $2 \tan 3x = 8$

44. $\frac{1}{2} \sin 3x = \frac{1}{4}$

45. Show that in any right triangle ABC $\tan \theta = \dfrac{\sin \theta}{\cos \theta}$ is true when angle θ is angle B.

Chapter 1 summary

- A degree can be divided into smaller parts in one of two ways—into minutes and seconds or into decimal parts.
- The sum of the measures of the three angles of any triangle is 180°.
- A right triangle is a triangle in which one of the angles is a right (90°) angle. The side of a right triangle that is opposite the right angle is called the hypotenuse, and the sides that form the right angle are called the legs.
- The standard method of labeling right triangles is to label the right angle C and the two acute angles A and B. The sides are always labeled a and b, with a opposite angle A and b opposite angle B. The hypotenuse is always labeled c.
- **The Pythagorean theorem** Given a right triangle with legs a and b and hypotenuse c, then $a^2 + b^2 = c^2$.
- If $a^2 + b^2 = c^2$ in a triangle, then that triangle is a right triangle, and the right angle is opposite side c.
- If θ is one of the two acute angles in a right triangle, then

$$\sin \theta = \frac{\text{length of leg opposite } \theta}{\text{length of hypotenuse}}$$

$$\cos \theta = \frac{\text{length of leg adjacent to } \theta}{\text{length of hypotenuse}}$$

$$\tan \theta = \frac{\text{length of leg opposite } \theta}{\text{length of leg adjacent to } \theta}$$

$$\sec \theta = \frac{1}{\cos \theta}, \ \csc \theta = \frac{1}{\sin \theta}, \ \cot \theta = \frac{1}{\tan \theta}$$

- **Fundamental identity of trigonometry** If θ is either acute angle in a right triangle, then

$$\sin^2\theta + \cos^2\theta = 1$$

- The following pairs of ratios are reciprocals:
  - cosine and secant
  - sine and cosecant
  - tangent and cotangent
- If we know one of the trigonometric ratios of an angle, we can find a triangle for which that ratio is true. We can use this triangle to compute the other five trigonometric ratios for that angle.
- The value of a trigonometric ratio depends only on the measure of the angle and not on the right triangle in which it appears.
- A right triangle is said to be solved once the lengths of its three sides and the measures of its two acute angles are known. To solve a right triangle, we must already know at least one side and either another side or one acute angle.
- **Several important trigonometric identities** are

$$\sin \theta = \frac{1}{\csc \theta}, \ \cos \theta = \frac{1}{\sec \theta}, \ \tan \theta = \frac{1}{\cot \theta},$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta}, \ \cot \theta = \frac{\cos \theta}{\sin \theta}$$

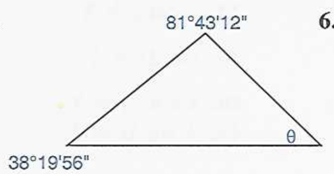
Chapter 1 review

[1–1] Convert each angle to its measure in decimal degrees. Round your answer to the nearest 0.01°. Also, state whether each angle is acute, obtuse, or right.

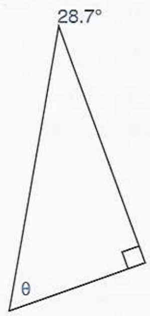
1. 17°34′17″ **2.** 84°9′ **3.** 125°37′ **4.** 39°45′43″

Find the measure of the missing angle, θ.

5.



6.

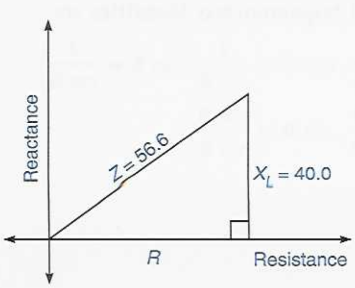


In the following problems two of the three sides of a right triangle are given. Use the Pythagorean theorem to calculate the length of the missing side; give your answer in exact form and in approximate form, rounded to one decimal place if necessary.

	a	b	c
7.	8	12	?
8.	9	?	26
9.	?	8	13

10. A flagpole is 46 feet tall and is supported by a wire attached at the top of the pole and to the ground 25 feet from the base of the pole. How long is the wire (to the nearest foot)?

11. The diagram is called an impedance diagram and is used in computing total impedance in a certain electronics circuit. It shows that inductive reactance X_L is 40.0 ohms and that impedance Z is 56.6 ohms. Calculate the resistance R to the nearest tenth of an ohm.



[1–2] In the following problems you are given parts of right triangle ABC. Use this information to compute the six trigonometric ratios for the angle specified.

	a	b	c	Find ratios for this angle
12.	3	7		A
13.	5		15	B
14.	2	$\sqrt{10}$		A

15. In right triangle ABC, $\sin A = \frac{4}{5}$. Draw a triangle for which this is true and use it to find $\sin B$.

16. In right triangle ABC, $\tan A = 5$. Draw a triangle for which this is true and use it to find $\cos B$.

17. In right triangle ABC, $\sec A = 3$. Draw a triangle for which this is true and use it to find $\tan B$.

18. In right triangle ABC, $\tan B = 0.8$. Draw a triangle for which this is true and use it to find $\sec A$.

[1–3] Use an electronic calculator to find four-decimal-place approximations for the following.

19. sin 15.5° **20.** sec 68.2° **21.** tan 17.9°
22. sin 40.8° **23.** tan 16.3° **24.** sec 25.7°
25. cot 31°20′ **26.** sec 85°40′ **27.** tan 31°30′
28. csc 43°10′ **29.** tan 63°30′ **30.** sec 86°40′

31. Find four-digit approximations for all six trigonometric ratios for
a. 53.20° **b.** 53°20′

32. A surveyor needs to compute R in the following formula as part of finding the area of the segment of a circle. $R = \dfrac{LC}{2 \sin I}$. Find R to three decimal places if $LC = 315.2$ meters and $I = 22.6°$.

Find the unknown acute angle θ to the nearest 0.1°.

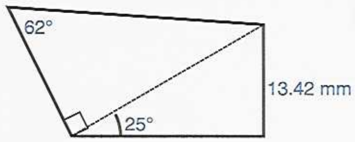
33. $\sin \theta = 0.6314$ **34.** $\cos \theta = 0.1382$
35. $\tan \theta = \dfrac{35.9}{50.0}$ **36.** $\sec \theta = 2.351$
37. $\csc \theta = 1.425$ **38.** $\cot \theta = 6.310$

In the following problems two of the five values that must be known to solve right triangle ABC are given. Find the other three values. Round all answers to the same accuracy as the data.

	a	b	c	A	B
39.	23.3			56.1°	
40.		11.4	23.0		
41.	4.00	8.55			
42.			66.0		63.1°

43. Ground radar shows that an aircraft is 22.6 kilometers from the radar site, at an angle of elevation of 11.2°. Find the aircraft's altitude to the nearest 0.01 kilometer.

44. The diagram is a top view of a portion of a machine part. Find the length x to the nearest tenth of a millimeter.



[1–4]

45. Show that the fundamental identity of trigonometry is true for an angle of measure 45°. (Use exact values.)

Simplify the following trigonometric expressions:

46. $\cot\theta\sec\theta$

47. $\csc\theta(\sin\theta - \tan\theta)$

48. $\dfrac{\sin\theta - 1}{\sin\theta}$

49. $\dfrac{\cos\alpha + 2 - \cot\alpha}{\cos\alpha}$

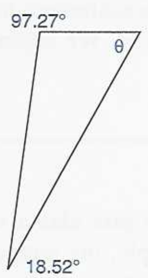
50. $(1 + \sin\theta)(1 - \sin\theta)$

51. Solve $3\sin 2x = 2$ for x to the nearest 0.1°.

Chapter 1 test

1. Convert 26°27′43″ to its measure in decimal degrees. Round your answer to the nearest 0.001°.

2. Find the measure of the missing angle θ.



3. In right triangle ABC, $a = 6$ and $c = 12$. Find b. Leave your answer in exact form.

4. A flagpole is 46 feet tall and is supported by an 87-foot guy wire attached at the top of the pole and to the ground. How far from the base of the pole is the ground attachment point of the wire (to the nearest 0.1 foot)?

5. In right triangle ABC, $\sec A = \frac{5}{3}$. Draw a triangle for which this is true and use it to find $\tan B$.

Find approximations for the following, to four decimal places.

6. $\sin 25.5°$

7. $\sec 78.3°$

8. $\cot 31°50'$

9. The average power, in watts, in an AC circuit is given by the formula $P = VI\cos\theta$. Compute P (in watts) if $V = 220$ volts, $I = 4.1$ amperes, and $\theta = 48°$ (to the nearest 0.1 watt).

Find the unknown acute angle θ to the nearest 0.1°.

10. $\sin\theta = 0.1314$

11. $\sec\theta = 4.121$

12. In right triangle ABC, $a = 3.8$ and $A = 21.4°$. Solve the triangle, rounding answers to the nearest tenth.

13. In right triangle ABC, $a = 5.2$ and $b = 7.9$. Solve the triangle, rounding answers to the nearest tenth.

14. If the angle of depression from an aircraft to a ground point 13.6 miles away is 12.5°, how high is the aircraft flying, to the nearest ten feet?

15. Simplify the expression $\cos\theta(\sec\theta - \cos\theta)$.

16. Solve $\sec 5x = 5$ for x to the nearest 0.1°.

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